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The Research on the L(2,1)-labeling problem from Graph theoretic and Graph Algorithmic Approaches

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Abstract

Graph coloring is one of the most popular topics in graph theory and many advances in graph theory are a direct consequence of graph coloring research. The $L(2,1)$-labeling problem is a generalization of the vertex coloring problem and its application background is the frequency assignment problem. The $L(2,1)$-labeling problem has been extensively researched on many graph classes. In this thesis, we have also studied the problem on some particular classes of graphs.

In Chapter 2 we obtain upper bounds for $L(2,1)$-labeling numbers of the four standard graph products and get significant improvements over the previously best known bounds for them.

In Chapter 3 we study the $L(2,1)$-labeling number of the composition of n graphs. We show that the $L(2,1)$-labelling number of the composition of n graphs is much smaller than the square of the maximum degree.

In Chapter 4 we consider the Cartesian sum of graphs and derive, both, lower and upper bounds for their $L(2,1)$-labeling number. We use two different approaches to derive the upper bounds and both approaches improve previously known bounds. We also present new approximation algorithms for the $L(2,1)$-labeling problem on Cartesian sum graphs.

In Chapter 5 we characterize d-disk graphs for $d > 1$, and give the first upper bounds on the $L(2,1)$-labeling number for this class of graphs.

In Chapter 6 we compute upper bounds for the $L(2,1)$-labeling number of total graphs of $K_{1,n}$-free graphs, where $K_{1,n}$ is the complete bipartite graph with one vertex in one side of the partition and n in the other.

In Chapter 7 we obtain more results on $L(2,1)$-labelings of the four standard graph products.

In Chapter 8 we determine the exact value for the $L(2,1)$-labeling number of a particular class of *Mycielski* graphs. We also provide, both, lower and upper bounds for the $L(2,1)$-labeling number of any *Mycielski* graph.

Key words: Graph coloring, frequency assignment, $L(2,1)$-labeling, Cartesian product, composition, direct product, strong product, Cartesian sum, d-disk graphs, total graphs, $Mycielski$ graphs.

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Chapter 1

Introduction

1.1 Basic Concepts and Notations

In this section we explain some concepts and notations that we use in this thesis. For a real number a, we denote $\lfloor a \rfloor$ as the largest integer which is not greater than a and $\lceil a \rceil$ as the smallest integer which is not less than a. For a set S, we denote $|S|$ as the total number of elements in S.

A *graph* G is an ordered tuple $(V(G), E(G))$, where $V(G)$ is an non-empty set of vertices, and $E(G)$ is a set of edges. An edge $e = (u, v) \in E(G)$ is said to join the two vertices u, v; u, v are called the *ends* of e. Sometimes we denote an edge (u, v) also as uv. G is called an *undirected graph* if there is no order between the two vertices u, v of an edge (u, v), so $(u, v) = (v, u)$; otherwise, G is called a *directed graph*. See Figure 1.1 for an example of an undirected graph and Figure 1.2 for an example of a directed graph. Usually, we simply denote a graph as $G = (V, E)$. An edge is called a *loop* if its two ends are identical; otherwise, it is called a *link*. Edge e_7 in Figure 1.1 is an example of loop. Two edges e_1 and e_2 are called *multiple edges* if they have the same endpoints. For example, e_1 and e_2 in Figure 1.1 are multiple edges. G is called a *simple graph* if it has no loops or multiple edges. We usually talk about simple graphs in this thesis unless specifically stated otherwise.

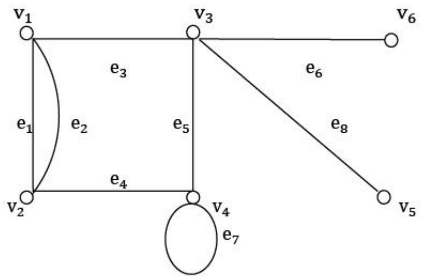

Fig. 1.1: An undirected graph.

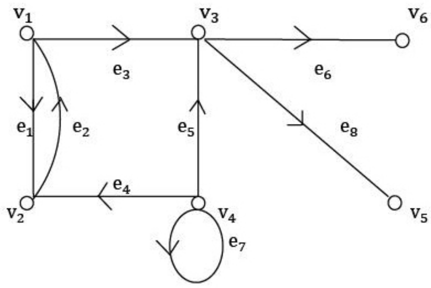

Fig. 1.2: A directed graph.

The two ends of an edge are said to be *incident* on the edge. Two vertices are called *adjacent* if they are incident on a common edge. The set of vertices which are adjacent to a vertex u is called the *neighborhood* of u and is denoted as $N_G(u)$. The *degree* of u is defined as $|N_G(u)|$ and denoted as $d_G(u)$. The *maximum degree* of G, denoted $\Delta(G)$, is the largest degree among all vertices in G and the *minimum degree* of G, denoted (G), is the smallest degree among all vertices in G. The vertices with degree 0 are called *isolated vertices*. For example, the maximum degree of the graph in Figure 1.1 is 4 and the minimum degree is 1.

Let $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ such that for every edge $(u, v) \in E(H)$, $u \in V(H)$ and $v \in V(H)$, then $H = (V(H), E(H))$ is called a *subgraph* of G, denoted $H \subseteq G$. If $H \subseteq G$ and $V(H) = V(G)$, then H is called a *spanning subgraph* of G. Given a graph $G = (V, E)$, for a nonempty subset V' of V, the subgraph which has V'

as its vertex set and all the edges whose two ends are in V' as its edge set is called the *subgraph induced by* V', denoted $G[V']$. For a nonempty subset E' of E, the subgraph which has E' as its edge set and the set of ends of edges in E' as its vertex set is called the subgraph of G induced by E', denoted $G[E']$.

Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be two graphs; the graph $G = (V_1 \bigcup V_2, E_1 \bigcup E_2)$ is called the *union* of G_1 and G_2 and the graph $G = (V_1 \bigcap V_2, E_1 \bigcap E_2)$ is called the *intersection* of G_1 and G_2.

A *complete graph* is a simple graph such that any two different vertices are adjacent. A complete graph with n vertices is denoted as K_n. See Figure 1.3 for an example of a complete graph. Let V_1, V_2 be two subsets of the vertex set of G such that $V_1 \bigcup V_2 = V(G)$, $V_1 \bigcap V_2 = \phi$, and for which one end of each edge in G is in V_1 and the other is in V_2, then G is called a *bipartite graph*, denoted as $G = (V_1 \bigcup V_2, E)$. See Figure 1.4 for an example of a bipartite graph. Given a bipartite graph $G = (V_1 \bigcup V_2, E)$, if every vertex in V_1 is adjacent to every vertex in V_2, then G is called a *complete bipartite graph*. A complete bipartite graph $G = (V_1 \bigcup V_2, E)$ with $|V_1| = p, |V_2| = q$ is denoted as $K_{p,q}$. See Figure 1.5 for an example of a complete bipartite graph. If a graph G can be drawn such that its edges only intersect at their ends, then G is called a *planar graph*; otherwise, it is called a *non-planar graph*.

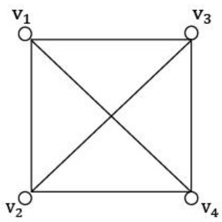

Fig. 1.3: Complete graph K_4.

A *walk* in G is a finite non-empty sequence $w = v_0 e_1 v_1 e_2 ... e_k v_k$ of alternating vertices and edges such that for $1 \leq j \leq k$, the ends of e_j are v_{j-1} and v_j. The

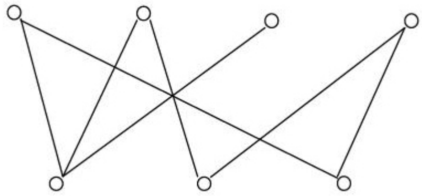

Fig. 1.4: A bipartite graph.

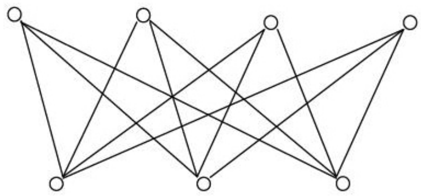

Fig. 1.5: Complete bipartite graph $K_{4,3}$.

vertices v_0, v_k are called *origin* and *terminus* and integer k is called the *length* of w. For example, in Figure 1.6, $v_1e_1v_2e_2v_1e_3v_3e_8v_5e_9v_4e_5v_3e_3v_1e_1v_2$ is a walk. If all the edges $e_1, e_2, ..., e_k$ of w are distinct, then w is called a *trail*. For example, in Figure 1.6, $v_1e_1v_2e_2v_1e_3v_3e_8v_5e_9v_4$ is a trail. If all the vertices $v_1, v_2, ..., v_k$ of a trail w are distinct, then w is called a *path*. For example, in Figure 1.6, $v_2e_1v_1e_3v_3e_8v_5e_9v_4$ is a path.

If a walk w has positive length and its origin and terminus are the same, then it is called *closed*. If the origin and internal vertices of a closed walk w are distinct, then w is called a *cycle*. For example, in Figure 1.6, $v_1e_3v_3e_5v_4e_4v_2e_1v_1$ is a cycle. If the length of a cycle is odd, then it is called an *odd cycle*; otherwise, it is called an *even cycle*. A *wheel* consists of a cycle and one vertex which is adjacent to each vertex

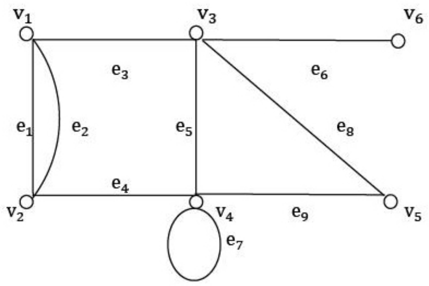

Fig. 1.6: A graph.

of the cycle. The *girth* of a graph G is the length of the shortest cycle in G. For example, the graph in Figure 1.7 is of girth 3.

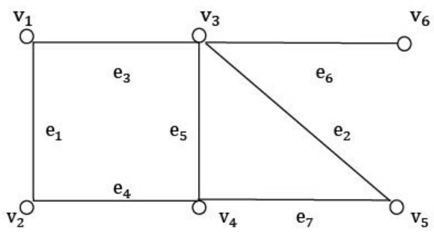

Fig. 1.7: A graph of girth 3.

If there is a path between vertices u and v, then u and v are said to be *connected* and the length of the shortest path between u and v is called the *distance* between u and v. Connection is an equivalent relationship on the vertex set V. There is a partition $V_1, V_2, ..., V_b$ of V such that any two vertices are connected if and only if they belong to the same partition V_k. The subgraphs $G[V_1], G[V_2], ..., G[V_b]$ are called *components* of G. If $b = 1$, then G is connected.

A subset S of $V(G)$ is called an *independent set* if and only if no two vertices in S are adjacent in G. If there is no independent set S' in G such that $|S'| > |S|$, then S is called a *maximum independent set.* For example, in Figure 1.7, $\{v_1, v_5, v_6\}$ is a maximum independent set. The number of vertices in a maximum independent set is called the *independence number* of G and is denoted as $\alpha(G)$.

A *k-vertex proper coloring* of G is an assignment of k colors $1, 2, ..., k$ to the vertices of G such that no two adjacent vertices have the same color. G is *k-vertex proper colorable* if it has a k-vertex proper coloring. The *vertex chromatic number* of G, $\chi(G)$, is the minimum k such that G is k-vertex proper colorable. For example, the vertex chromatic number of the graph in Figure 1.8 is 3 and the figure shows a 3-vertex proper coloring.

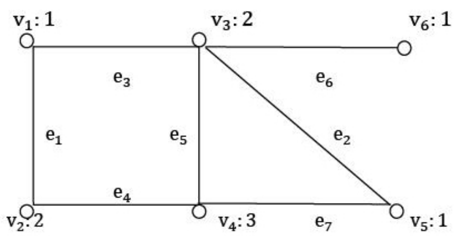

Fig. 1.8: A coloring for the vertices of the graph in Figure 7.

For a subset S of $V(G)$, if $G[S]$ is a complete graph, then S is called a *clique* of G.

The *adjacency matrix* of a graph G of n vertices is a $n \times n$ matrix $A = (a_{ij})$ where the non-diagonal entry a_{ij} $(i \neq j)$ is the number of edges from vertex i to vertex j, and the diagonal entries are all equal to zero.

Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be a $p \times q$ matrix, the *Kronecker product* $A \otimes B$ is the $mp \times nq$ block matrix

$$\begin{pmatrix} a_{11}B & ... & a_{1n}B \\ ... & ... & ... \\ a_{m1}B & ... & a_{mn}B \end{pmatrix}$$

1.2 Coloring Problems

Graph coloring is one of the most popular topics in graph theory and many advances in graph theory have been a direct consequence of graph coloring research. The study of graph coloring problems can be traced to over one hundred years ago, as these problems have a large number of applications. For example, the vertex coloring problem consists in assigning colors to the vertices of a graph so that adjacent vertices are assigned different colors. This problem can be used, for example, to model scheduling and register allocation problems. In a scheduling problem the goal is to assign a set of jobs to a group of machines so as to minimize some objective function, like the time needed to process them all. Consider, for example, a scheduling problem where jobs have unit length and there are several pairs of conflicting jobs which cannot be processed by the same machine. A vertex coloring model for this scheduling problem can be formulated as follows. Consider a graph whose vertices represent the jobs and in which there is an edge between any two vertices if the corresponding two jobs conflict. The minimum time for the jobs to complete satisfying the conflict restrictions is just the vertex chromatic number of the above graph. See Figure 1.9 for an example of this scheduling problem.

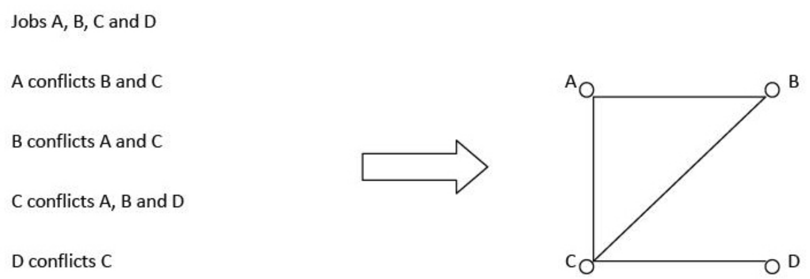

Fig. 1.9: A set of 4 jobs A, B, C, D with conflicting pairs $(A, B), (A, C), (B, C), (C, D)$ and its corresponding graph coloring model.

The register allocation problem is to assign a number of variables to a set of registers; each variable needs to be kept in a register for only a limited amount of time, and the goal is to find the minimum number of registers needed to store all the variables. A vertex coloring model can be formulated as follows. Build a graph whose vertices represent the registers and in which there is an edge between any two vertices if the corresponding two registers are needed to store variables at the same time. The minimum number of registers needed is just the chromatic number of this graph. See Figure 1.10 for an example of the register allocation problem.

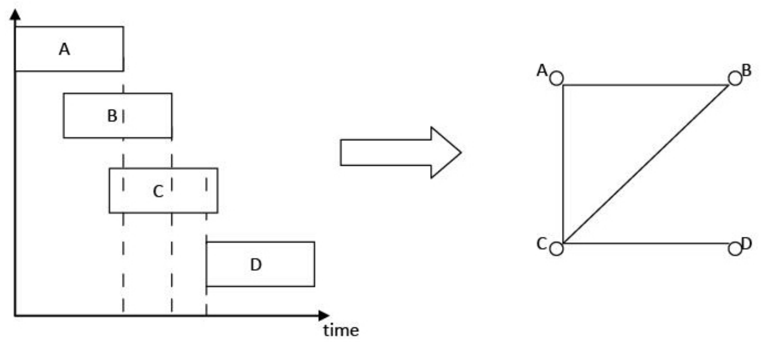

Fig. 1.10: An instance of the register allocation problem and its corresponding graph coloring model. Four variables, A, B, C, and D need to be stored in the registers. When two variables have to be stored at the same time, they conflict.

1.3 The $L(2,1)$-Labeling Problem: A Graph Theoretic Model for the Frequency Assignment Problem

The $L(2,1)$-labeling problem is a generalization of the vertex coloring problem and so it has many applications and it has been extensively studied. An $L(2,1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if x and y are adjacent and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$, where $d(x, y)$ denotes the distance between x and y in G. The $L(2, 1)$-labeling number of G, denoted by $\lambda(G)$, is the smallest number k such that there is an L(2,1)-labeling with maximum label k. An L(2,1)-labeling having maximum label λ is called optimal. For example, the $L(2, 1)$-labeling number of the graph in Figure 1.11 is 6 and 6-$L(2, 1)$-labeling is shown.

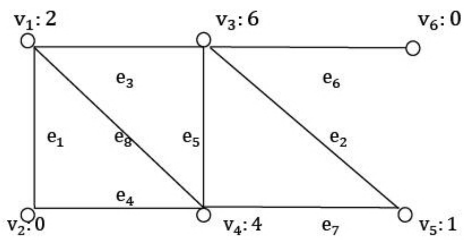

Fig. 1.11: An $L(2, 1)$-labeling for the graph in Figure 7.

The $L(2, 1)$-labeling problem naturally arises from the frequency assignment problem in wireless networks. In the frequency assignment problem we are given a number of transmitters or stations and we need to assign a frequency to each one of them so that transmitters do not interfere with each other. In practice it makes sense to consider two levels of interference: (1) two ‘very close’ transmitters between which very strong interference may occur must receive frequencies that differ by at least two channels, and (2) two ’close’ transmitters must receive different frequencies. The definition of ’very close’ and ’close’ depends on the physical characteristics of the transmitters. This problem can be modelled by a graph in which assigning frequencies to the transmitters is equivalent to assigning an L(2,1)-label to each vertex. We seek the smallest difference between the highest and lowest frequencies assigned, to minimize the total bandwidth used. Over 100 references on the $L(2,1)$-labeling problem are provided in a very comprehensive survey [6] by Calamoneri. Due to the inherent hardness of $L(2,1)$-labeling problems, most of these papers consider only particular classes of graphs. From the algorithmic point of view it is not surprising that it is NP-complete to decide whether a given graph G allows an $L(2,1)$-labeling with maximum label $\lambda(G)$ [22]. Hence good lower and upper bounds for $\lambda(G)$ are clearly welcome. For instance, if G is a diameter 2 graph, then $\lambda(G) \leq \Delta^2$ where Δ is the maximum degree of G. This upper bound is attainable by *Moore graphs* (diameter 2 graphs with $\Delta^2 + 1$ vertices), see [22]; such graphs exist for $\Delta = 2, 3, 7$, and possibly 57.

In 1992 Griggs and Yeh [22] conjectured that for any graph G with maximum degree $\Delta \geq 2$, $\lambda(G) \leq \Delta^2$. Note that this is not true for $\Delta = 1$ since for example, $\Delta(K_2) = 1$ but $\lambda(K_2) = 2$. Griggs and Yeh [22] proved that $\lambda \leq \Delta^2 + 2\Delta$ for general graphs with maximum degree Δ. Chang and Kuo [10] improved the bound to $\Delta^2 + \Delta$, and then Král' and Škrekovski [34] further reduced the bound to $\Delta^2 + \Delta - 1$.

Approximation algorithms and inapproximability results for the $L(2,1)$-labeling problem are rare. In [7], by applying an algorithm by McCormick [40], Calamoneri et al. proved that there is an algorithm for the $L(2,1)$-labelling problem with approximation ratio $2((n-1)^{1/2} + 1)$, where n is the number of vertices in the input graph. In [24], Halldorsson improved the above result by proving that the approximation ratio of the first-fit algorithm is $\min\{n^{1/2} + 2, \Delta\}$ which is the currently best known result. He also proved that it is hard to approximate the $L(2,1)$-labelling problem within a factor of $n^{1/2-\varepsilon}$ for any $\varepsilon > 0$.

In this thesis, we present some of our results in this area and we propose some interesting open problems. Throughout the document, all graphs are assumed to be simple (i.e. they have no loops or multiple edges).

1.4 The Graph Classes Studied in this Thesis

In this section, we define the various graph classes researched in this thesis. The *Cartesian product* of two graphs G and H is the graph $G \square H$ with vertex set $V(G) \times V(H)$, in which a vertex (v, w) is adjacent to a vertex (v', w') if and only if either $v = v'$ and w is adjacent to w' in H or $w = w'$ and v is adjacent to v' in G. See Figure 1.12 for an example of the Cartesian product of two graphs.

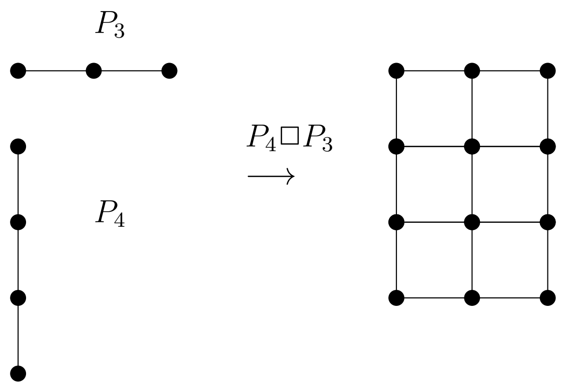

Fig. 1.12: Cartesian product of graphs P_3 and P_4.

The *composition* (or *lexicographic product*) of two graphs G and H is the graph $G[H]$ with vertex set $V(G) \times V(H)$, in which a vertex (u, v) is adjacent to a vertex (u', v') if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. See Figure 1.13 for an example of the composition of two graphs.

The *direct product* $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which a vertex (v, w) is adjacent to a vertex (v', w') if and only if v is adjacent to v' in G and w is adjacent to w' in H. See Figure 1.14 for an example of the direct product of two graphs.

The *strong product* $G \boxtimes H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which a vertex (v, w) is adjacent to a vertex (v', w') if and only if $v = v'$

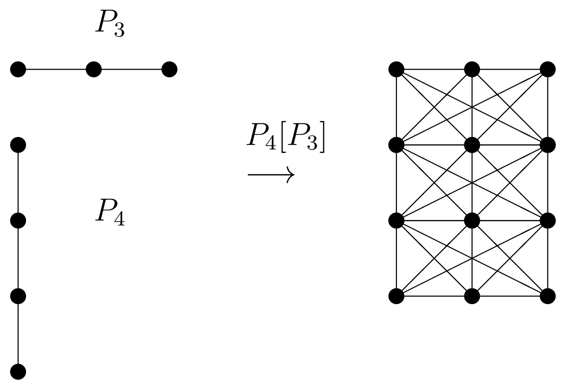

Fig. 1.13: Composition of graphs P_3 and P_4.

and w is adjacent to w' in H, or $w = w'$ and v is adjacent to v' in G, or v is adjacent to v' in G and w is adjacent to w' in H. See Figure 1.15 for an example of this graph product.

The *Cartesian sum*, $G \bigoplus H$, of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which a vertex (u, v) is adjacent to another vertex (u', v') if and only if either $uu' \in E(G)$, or $vv' \in E(H)$, or both [44]. See Figure 1.16 for an example of the Cartesian sum of two graphs.

Consider some product $G \bullet H$ of two graphs G and H. Graphs G and H called the *factors* of the product.

A *d-sphere*, $d \geq 2$, is the set of points $(x_1, x_2, \cdots, x_d)$ in R^d such that $(x_1 - c_1)^2 + (x_2 - c_2)^2 + \cdots + (x_d - c_d)^2 = r^2$, where r is the *radius* and $(c_1, c_2, \cdots, c_d) \in R^d$ is the center of the d-sphere. The 2-sphere and 3-sphere are the usual *circle* and *sphere*, respectively. The *diameter* of a sphere of radius r is $2r$. A d-sphere with diameter one is called a *unit d-sphere*.

A graph G is called a *unit d-disk graph*, if we can assign a unit d-sphere to each vertex of G so that two vertices are adjacent if and only if the corresponding spheres

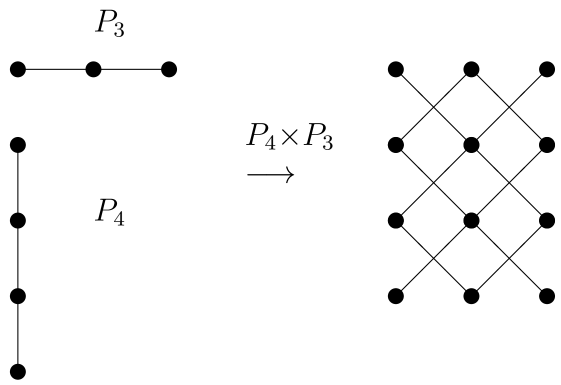

Fig. 1.14: Direct product of graphs P_3 and P_4.

overlap. The set D of spheres assigned to the vertices of G is called the disk representation of G.

Example. The right side of Figure 1.17 shows a 2-disk graph called *the triangular lattice graph*, $\Gamma(\Delta)$, and its disk representation is shown on the left side of Figure 1.17. $\Gamma(\Delta)$ is an infinite graph and it is $K_{1,4}$-free.

Let D be the disk representation of a d-disk graph G. Let d_{min} and d_{max} be the minimum and maximum diameters of the d-spheres in D. The value d_{max}/d_{min} is called the *diameter ratio* of D, denoted by $\sigma(D)$. A disk graph G is called a $\sigma(D)$-disk graph if it has a disk representation D of diameter ratio $\sigma(D)$. If $\sigma(D) = 1$, then G is a unit d-disk graph; in this case, we assume that all spheres in D have unit diameter.

Given an undirected graph $G = (V, E)$, the *total graph* $T(G) = (V', E')$ of G is the undirected graph with vertex set $V' = V \bigcup E$ and edge set $E' = \{(u, v)|(u, v) \in E$, or $u = (t, t') \in E$ and $v = (t', p) \in E$, or $v = (u, t') \in E\}$. See Figure 1.18 for an example of a total graph.

Let G be a simple graph having vertex set $\{v_1, \cdots, v_n\}$. The *Mycielski graph* $\mu(G)$ is the graph obtained from G by adding to it a vertex w, vertices $U = \{u_1, \cdots, u_n\}$

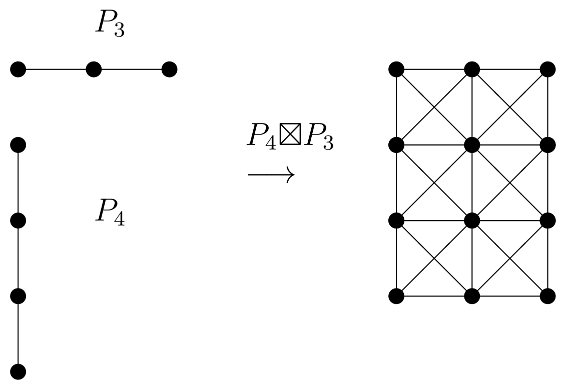

Fig. 1.15: Strong product of graphs P_3 and P_4.

and edges $\{(u_i, w), (u_i, v_j) | i, j = 1, ..., n, i \neq j\}$. See Figure 1.19 for an example of a Mycielski graph. Furthermore, we can recursively define $\mu^t(G) = \mu(\mu^{t-1}(G))$, for $t \geq 2$. It is well known that if G is a triangle-free (K_3-free) graph then $\mu(G)$ is also triangle-free [64].

1.5 Related Work

The $L(2,1)$-labeling problem has been extensively studied on many graph classes. We have also studied the problem on several particular classes of graphs. The graph classes that we have studied are either models for real networks and thus, they have many potential applications, or they are of theoretical importance. For example, paths, cycles, and cliques model buses, rings, and mesh networks; these are the simplest and most common networks. The classes of total graphs and Mycielski graphs are important graph classes widely used as benchmarks in graph coloring problems.

Because many interesting wireless networks have simple factors, such as paths and cycles and we can gain global information about a network from its factors, product graphs have been extensively considered. For example, an n-dimensional grid is the

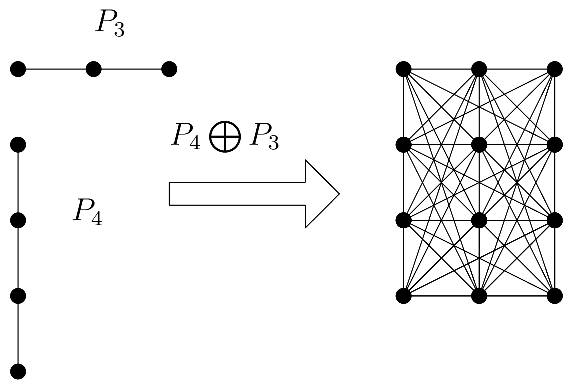

Fig. 1.16: Cartesian sum of graphs P_3 and P_4.

Cartesian product of n paths, an n-dimensional torus is the Cartesian product of n cycles, a Hamming graph is the Cartesian product of n copies of the complete graph K_2, and an octagonal grid is the strong product of two paths.

Graph products play an important role in defining various useful types of networks and they also serve as natural tools for studying different concepts in many areas of research. For example, one of the central concepts of information theory, the Shannon capacity, is most naturally expressed with the strong product of graphs [62].

From the viewpoint of human epistemology, it is customary to first devote research to fundamental concepts and then gradually develop more complex ideas. Research on the $L(2,1)$-labeling problem also underwent such a process. For example, researchers first explored labelings on fundamental structures such as paths, circles, and wheels. Then they focused their research on increasingly more complex structure like Cartesian product, composition or lexicographic product, direct product, strong product, Cartesian sum product of graphs, and so on.

Whittlesey et al. [65] considered $L(2,1)$-labelings of the Cartesian products of paths. $L(2,1)$-labelings for the Cartesian product of a path and a cycle as well as the Cartesian product of two or more cycles have been studied in several articles

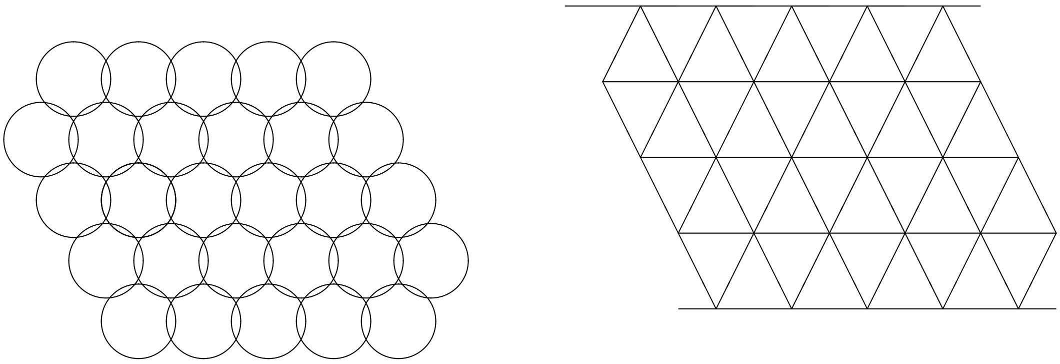

Fig. 1.17: The triangular lattice graph $\Gamma(\Delta)$, and its disk representation.

([27], [28], [30], [32], [36] and [47]). $L(2,1)$-labeling for the Cartesian product of complete graphs has also been considered by Georges et al. [20]. Shao and Yeh [53] proved that Griggs and Yeh's conjecture is true for the Cartesian product and the composition of any two graphs (with minor exceptions).

Jha, Klavžar and Vesel [29] considered the direct product of a path and a cycle and the direct product of two cycles and got bounds for their $L(2,1)$-labeling numbers. Jha [28] considered the $L(2,1)$-labeling problem on the strong product of k cycles. Klavžar and Spacapan [31] proved that Griggs and Yeh's conjecture is true for the direct and strong products of any two graphs. Shao et al. [49] later obtained improved upper bounds for the $L(2,1)$-labeling number of the direct and strong products of any two graphs through the use of a refined combinatorial analysis.

Shao and Zhang [57] proved that Griggs and Yeh's conjecture is true for the Cartesian sum of any two graphs.

Unit interval graphs and their generalization, the class of unit d-disk graphs, are of particular interest in the frequency assignment problem. Given a set of transmitters,

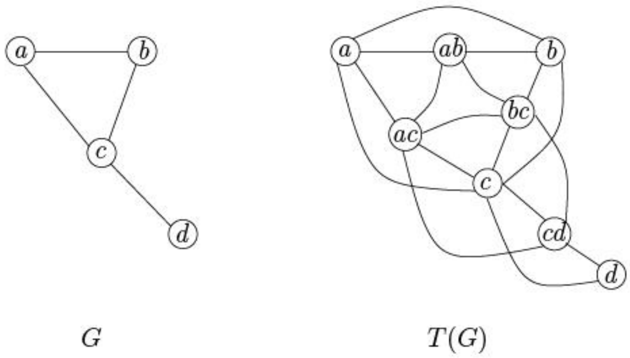

Fig. 1.18: A graph G and its corresponding total graph.

interference can take place if two transmitters are within a certain distance from each other. We can model this situation with the so-called interference graphs, which for this particular example are unit d-disk graphs. Unfortunately, no useful characterization of unit d-disk graphs was known except for $d = 1$. Using this characterization, Sakai [46] gave upper bounds for the $L(2, 1)$-labeling number of unit 1-disk graphs and proved that these upper bounds corroborate the conjecture of Griggs and Yeh. Shao et al. [55] characterized unit d-disk graphs for $d = 2, 3$ and gave upper bounds for the $L(2, 1)$-labeling number for this class of graphs.

1.6 Our Contributions

We have several results on $L(2, 1)$-labeling problems, which have been published or have been accepted for publication in reputed scientific journals. We also have some other work in progress in this area. In the next chapters we present our main results in this field.

We have obtained some results on product graphs. We have determined both lower and upper bounds for the Cartesian sum of any two graphs and for the composition

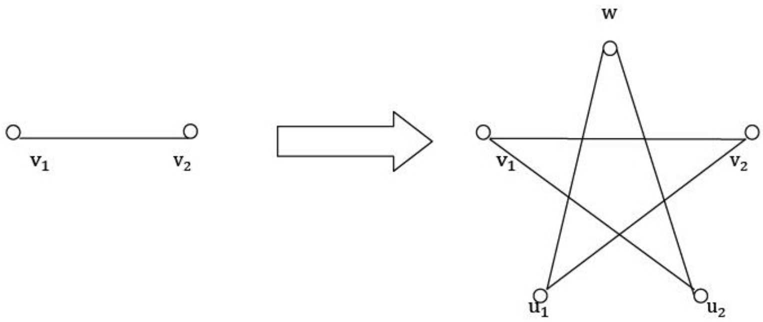

Fig. 1.19: The Mycielski graph $\mu(K_2)$.

product of n graphs and these bounds improve on previously known bounds for these classes of graphs. We also designed approximation algorithms for the Cartesian sum of any two graphs.

We have been able to characterize unit d-disk graphs for $d > 3$ and d-disk graphs for $d > 1$, and we found upper bounds on the $L(2, 1)$-labeling number for these two classes of graphs. We were able to show that for these graphs the conjecture of Griggs and Yeh is true only in some cases.

We computed upper bounds for the $L(2, 1)$-labeling number of total graphs of $K_{1,n}$-free graphs, where $K_{1,n}$ is the complete bipartite graph with one vertex in one side of the partition and n in the other. We also determined the exact value for the $L(2, 1)$-labeling number of a class of *Mycielski* graphs derived from complete graphs and provided lower and upper bounds for the $L(2, 1)$-labeling number of any *Mycielski* graph.

1.7 Organization of the Thesis

The structure of this thesis is as follows.

In Chapter 2 we present $L(2, 1)$-labelings for the product graphs. In Chapter 3 we present $L(2, 1)$-labelings for the composition of n graphs and in Chapter 4 we present, both, lower and upper bounds for the $L(2, 1)$-labeling number of Cartesian sum graphs. In Chapter 5 we present $L(2, 1)$-labelings of unit d-disk and d-disk graphs. In Chapter 6 we present $L(2, 1)$-labelings of total graphs and in Chapter 7 we present more results on $L(2, 1)$-labelings for the product graphs. In Chapter 8 we present $L(2, 1)$-labelings of Mycielski graphs.

In the last part of this thesis, we present our conclusions.

Chapter 2

$L(2,1)$-Labelings of Product Graphs

Graph products play an important role in connecting various useful networks and they also serve as natural tools for different concepts in many areas of research. For examples, the diagonal mesh with respect to multiprocessor network is representable by the direct product of two odd cycles [61] and one of the central concepts of information theory, the Shannon capacity, is most naturally expressed with the strong product of graphs, cf. [62].

The Cartesian product, the lexicographic product, the direct product and the strong product constitute the four standard graph products [25]. Shao and Yeh [53] proved that Griggs and Yeh's conjecture is true for the Cartesian product and the composition of any two graphs (with minor exceptions) and then Klavžar and Spacapan [31] proved that Griggs and Yeh's conjecture is true for the direct and strong products of any two graphs. Recently, Shao, Klavžar, Shiu and Zhang [49] improved the upper bounds obtained in [31] with a more refined analysis of neighborhoods in product graphs than the analysis in [31].

In this chapter, we study $L(2,1)$-labelings on the four standard graph products and obtain significant improvements over previously best results.

In the next section a heuristic labeling algorithm is presented that forms the basis for these considerations while in the remaining sections the four standard products of graphs are considered, respectively.

2.1 A Labeling Algorithm

A subset X of $V(G)$ is called an *i-stable set* (or *i-independent set*), if the distance between any two vertices in X is greater than i. A 1-stable set is a usual independent set. A *maximal* i-stable subset X of a set Y of vertices is an i-stable subset of Y such that X is not a proper subset of any other i-stable subset of Y. A maximal i-stable set of a give graph G can be computed by using a greedy algorithm: Pick any vertex of G and add it to the i-stable set; remove from G all vertices at distance at most i from the last vertex selected and then repeat the above procedure as long as G is not empty.

Chang and Kuo [10] proposed the following algorithm to compute an $L(2,1)$-labeling for a given graph.

Algorithm 2.1.1

Input: A graph $G = (V, E)$.
Output: The value k of the maximum label.

Idea: In each step, find a maximal 2-stable set from the unlabeled vertices that are at distance at least two from the vertices labeled in the previous step. Label all vertices in the 2-stable set with the index i of the current step. The index i starts from 0 and increases by 1 at each step. The maximum label k is the final value of i.

Initialization: Set $X_{-1} = \emptyset$; $V = V(G)$; $i = 0$.

Iteration:

1. Determine Y_i and X_i.

   - **If** $X_{i-1} \neq \emptyset$ **then** set $Y_i = \{x \in V : x$ is unlabeled and $d(x, y) \geq 2$ for all $y \in X_{i-1}\}$

else Set $Y_i = V$.

- **If** $Y_i \neq \emptyset$ **then** compute X_i, a maximal 2-stable subset of Y_i **else** set $X_i = \emptyset$.

2. Label the vertices in X_i (if there are any) with i.

3. $V \leftarrow V \backslash X_i$.

4. If $V \neq \emptyset$ then set $i \leftarrow i + 1$ and go to Step 1.

5. Record the current value of i as k (which is the maximum label). Stop.

Note that the value k computed by the above algorithm is an upper bound on $\lambda(G)$. We would like to find a bound for k in terms of the maximum degree $\Delta(G)$ of G, analogous to existing bounds for the chromatic number $\chi(G)$ in terms of $\Delta(G)$.

Let x be a vertex with the largest label k assigned by Algorithm 2.1.1. Denote

- $I_1 = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) = 1 \text{ for some } y \in X_i\}$. This is the set of labels of the neighbors of x.

- $I_2 = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) \leq 2 \text{ for some } y \in X_i\}$. This set consists of the labels of the vertices at distance at most 2 from x.

- $I_3 = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) \geq 3 \text{ for all } y \in X_i\}$. This set consists of the labels not used by vertices at distance at most 2 from x.

It is clear that $|I_2|+|I_3| = k$. For any $i \in I_3$, $x \notin Y_i$ since otherwise $X_i \cup \{x\}$ would be a 2-stable subset of Y_i, which contradicts the choice of X_i. That is, $d(x,y) = 1$ for some vertex y in X_{i-1}; i.e., $i-1 \in I_1$. Since for every $i \in I_3$, $i-1 \in I_1$, then $|I_3| \leq |I_1|$. Hence $k = |I_2| + |I_3| \leq |I_2| + |I_1|$.

In order to upper bound k, we will just find a bound for

$$B = |I_1| + |I_2| \tag{2.1}$$

in terms of $\Delta(G)$.

2.2 The Cartesian Product of Graphs

In [53], they obtained an upper bound on $\lambda(G\Box H)$ in terms of the maximum degree of $G\Box H$ for any two graphs G and H. In this section, we also consider this problem.

Theorem 2.2.1 *Let Δ_1 and Δ_2 be maximum degrees of G and H, respectively. Then $\lambda(G\Box H) \leq \Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2 + \Delta_1 + \Delta_2$.*

Proof. We first apply Algorithm 2.1.1 to label the graph $G\Box H$ and let k be the maximum label obtained by the algorithm. Let $x = (u, v)$ in $V(G)\times V(H)$ be a vertex with the label k. Then $deg_{G\Box H}(x) = deg_G(u) + deg_H(v)$. Denote $d = deg_{G\Box H}(x)$, $d_1 = deg_G(u)$, $d_2 = deg_H(v)$, $\Delta_1 = \Delta(G)$ and $\Delta_2 = \Delta(H)$. Hence $d = d_1 + d_2$ and $\Delta = \Delta(G\Box H) = \Delta_1 + \Delta_2$.

Let $G = (V, E)$ be a graph. Let A be its adjacency matrix with respect to the list of vertices $\{v_1, \ldots, v_n\}$. Then it is well-known that the (i, j)th entry of A^k is the number of different $(v_i - v_j)$-walks in G of length k, for $k \geq 0$. Thus, the number of the nonzero entries in the i-row of A^2 is the number of vertices of distance 2 from v_i (it includes the vertex v_i itself if $\deg(v_i) \neq 0$).

Let the order of G and H be ν_1 and ν_2, respectively. Suppose $V(G) = \{u_1, \ldots, u_{\nu_1}\}$ and $V(H) = \{v_1, \ldots, v_{\nu_2}\}$. Consider the cartesian product graph $G\Box H$. We list the vertex set $V(G) \times V(H)$ in lexicographic order. Then the adjacency matrix of $G\Box H$ with respect to this list is $A = A_1 \otimes I_2 + I_1 \otimes A_2$, where A_1 and A_2 are adjacency matrices of G and H respectively, I_1 and I_2 are the identity matrices of order ν_1 and ν_2 respectively. Note that $P \otimes Q$ is the Kronecker product of the matrices P and Q.

Then $A^2 + A = A_1^2 \otimes I_2 + 2A_1 \otimes A_2 + I_1 \otimes A_2^2 + A_1 \otimes I_2 + I_1 \otimes A_2$.

For fixed vertex (u_i, v_j) in $G\square H$, the number of nonzero entries in the (u_i, v_j)th row of $A^2 + A$ excluding the diagonal entries is the same as the number of nonzero entries in the (u_i, v_j)th row of $A_1^2 \otimes I_2 + A_1 \otimes A_2 + I_1 \otimes A_2^2 + A_1 \otimes I_2 + I_1 \otimes A_2$ excluding the diagonal entries.

For fixed vertex (u_i, v_j) in $G\square H$, we only look at the (u_i, v_j)th row of the above matrix. Then the number of nonzero entries in this row excluding the diagonal entries is at most

$\deg_G(u_i)(\Delta_1 - 1) + \deg_G(u_i)\deg_H(v_j) + \deg_H(v_j)(\Delta_2 - 1) + \deg_G(u_i) + \deg_H(v_j) = \deg_G(u_i)\Delta_1 + \deg_H(v_j)\Delta_2 + \deg_G(u_i)\deg_H(v_j)$.

Thus, $\lambda(G\square H) \leq |I_2| + |I_1| \leq \Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2 + \Delta_1 + \Delta_2$. ■

The above result agrees with the result in [53].

2.3 The Composition of Graphs

In [53], they provided an upper bound on $\lambda(G[H])$ in terms of the maximum degree of $G[H]$ for any two graphs G and H. In this section, we also consider this problem.

Theorem 2.3.1 *Let Δ_1 and Δ_2 be maximum degrees of G and H, respectively. Let ν_2 be the number of vertices of H. Then $\lambda(G[H]) \leq \Delta_1^2\nu_2 + \Delta_2^2 - 1 + \Delta_1\nu_2 + \Delta_2$.*

Proof. Again, we apply Algorithm 2.1.1 to obtain an $L(2,1)$-labeling with the maximum label k on the graph $G[H]$. Suppose $x = (u, v) \in V(G)\times V(H)(= V(G[H]))$ is labeled by k. Denote $d_1 = deg_G(u)$, $d_2 = deg_H(v)$, $\Delta_1 = \Delta(G)$, $\Delta_2 = \Delta(H)$ and $n = |V(H)|$. Then $d = deg_{G[H]}(x) = nd_1 + d_2$ and hence $\Delta = n\Delta_1 + \Delta_2$.

Let G and H be two graphs of order ν_1 and ν_2, respectively. Suppose $V(G) = \{u_1, \ldots, u_{\nu_1}\}$ and $V(H) = \{v_1, \ldots, v_{\nu_2}\}$. Consider the composition graph $G[H]$. We list the vertex set $V(G) \times V(H)$ in lexicographic order. Then the adjacency matrix of $G[H]$ with respect to this list is $A = A_1 \otimes J_2 + I_1 \otimes A_2$, where A_1 and A_2 are adjacency matrices of G and H respectively, J_2 is the square matrix of order ν_2 all of whose entries are equal to 1 and I_1 is the identity matrix of order ν_1. Note that $P \otimes Q$ is the Kronecker product of the matrices P and Q.

Then $A^2 + A = \nu_2 A_1^2 \otimes J_2 + A_1 \otimes J_2 A_2 + A_1 \otimes A_2 J_2 + I_1 \otimes A_2^2 + A_1 \otimes J_2 + I_1 \otimes A_2$ $= \nu_2 A_1^2 \otimes J_2 + A_1 \otimes (J_2 A_2 + A_2 J_2 + J_2) + I_1 \otimes A_2^2 + I_1 \otimes A_2$

For fixed vertex (u_i, v_j) in $G[H]$, the number of nonzero entries in the (u_i, v_j)th row of $A^2 + A$ excluding the diagonal entries is the same as the number of nonzero entries in the (u_i, v_j)th row of $A_1^2 \otimes J_2 + A_1 \otimes J_2 + I_1 \otimes A_2^2 + I_1 \otimes A_2$ excluding the diagonal entries.

Let Δ_1 and Δ_2 be the maximum degrees of G and H, respectively. For fixed vertex (u_i, v_j) in $G[H]$, we only look at the (u_i, v_j)th row of the above matrix. Then the number of nonzero entries in this row excluding the diagonal entries is at most

$\deg_G(u_i)(\Delta_1 - 1)\nu_2 + \deg_G(u_i)\nu_2 + \deg_H(v_j)(\Delta_2 - 1) + \deg_H(v_j) = \deg_G(u_i)\Delta_1\nu_2 + \deg_H(v_j)\Delta_2$.

Thus, $|I_2| + |I_1| \leq \Delta_1^2\nu_2 + \Delta_2^2 - 1 + \Delta_1\nu_2 + \Delta_2$. ■

In [53] it was proved that $\lambda(G[H]) \leq \Delta^2 + \Delta - 2\nu_2\Delta_1$.

Because $\Delta^2 + \Delta - 2\nu_2\Delta_1 - (\Delta_1^2\nu_2 + \Delta_2^2 - 1 + \Delta_1\nu_2 + \Delta_2) = \Delta_1^2\nu_2(\nu_2 - 1) + 2\Delta_1(\Delta_2 - 1)\nu_2$, we reduce the bound by $\Delta_1^2\nu_2(\nu_2 - 1) + 2\Delta_1(\Delta_2 - 1)\nu_2$.

2.4 The Direct Product of Graphs

In [31] and [49], they obtained upper bounds for the $L(2, 1)$-labeling number of the direct product of two graphs in terms of the maximum degree of $G \times H$ for any two graphs G and H. In this section, we also consider this problem.

Theorem 2.4.1 *Let Δ_1 and Δ_2 be maximum degrees of G and H, respectively. Then* $\lambda(G \times H) \leq \Delta_1^2\Delta_2^2 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 + 3\Delta_1\Delta_2$.

Proof. Again, we apply Algorithm 2.1.1 to obtain an $L(2,1)$-labeling with the maximum label k on the graph $G \times H$. Let $x = (u, v)$ in $V(G) \times V(H)$. Then $\deg_{G\times H}(x) = \deg_G(u)\deg_H(v)$. Denote $d = \deg_{G\times H}(x)$, $d_1 = \deg_G(u)$, $d_2 = \deg_H(v)$, $\Delta_1 = \Delta(G)$ and $\Delta_2 = \Delta(H)$. Hence $d = d_1d_2$ and $\Delta = \Delta(G \times H) = \Delta_1\Delta_2$.

Let G and H be two graphs of order ν_1 and ν_2, respectively. Suppose $V(G) = \{u_1, \ldots, u_{\nu_1}\}$ and $V(H) = \{v_1, \ldots, v_{\nu_2}\}$. Consider the direct product graph $G \times H$. We list the vertex set $V(G)\times V(H)$ in lexicographic order. Then the adjacency matrix of $G \times H$ with respect to this list is $A = A_1 \otimes A_2$, where A_1 and A_2 are adjacency matrices of G and H respectively. Note that $P \otimes Q$ is the Kronecker product of the matrices P and Q.

Then $A^2 + A = A_1^2 \otimes A_2^2 + A_1 \otimes A_2$. For fixed vertex (u_i, v_j) in $G \times H$, the number of nonzero entries in the (u_i, v_j)th row of $A^2 + A$ excluding the diagonal entries is the same as the number of nonzero entries in the (u_i, v_j)th row of $A_1^2 \otimes A_2^2 + A_1 \otimes A_2$ excluding the diagonal entries.

Let Δ_1 and Δ_2 be the maximum degrees of G and H, respectively. For fixed vertex (u_i, v_j) in $G \times H$, we only look at the (u_i, v_j)th row of the above matrix. Then the number of nonzero entries in this row excluding the diagonal entries is at most

$$\deg_G(u_i)(\Delta_1 - 1)\deg_H(v_j)(\Delta_2 - 1) + \deg_G(u_i)\deg_H(v_j).$$

Thus, $|I_2| + |I_1| \leq \Delta_1(\Delta_1 - 1)\Delta_2(\Delta_2 - 1) + 2\Delta_1\Delta_2 = \Delta_1^2\Delta_2^2 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 + 3\Delta_1\Delta_2$. ■

In [49] it was proved that $\lambda(G \times H) \leq \Delta^2 + \Delta - (\Delta_1 + \Delta_2)(\Delta_1 - 1)(\Delta_2 - 1)$.

Because $\Delta^2 + \Delta - (\Delta_1 + \Delta_2)(\Delta_1 - 1)(\Delta_2 - 1) - (\Delta_1^2\Delta_2^2 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 + 3\Delta_1\Delta_2) = \Delta_1^2 + \Delta_2^2 - \Delta_1 - \Delta_2$, we reduce the bound in [49] by $\Delta_1^2 + \Delta_2^2 - \Delta_1 - \Delta_2$.

2.5 The Strong Product of Graphs

In [28] the λ-numbers of the strong product of cycles were considered. In [31] and [49], they obtained upper bounds for the λ-number of strong products in terms of the maximum degree of $G \boxtimes H$ for any two graphs G and H. In this section, we also consider this problem.

Theorem 2.5.1 *Let Δ_1 and Δ_2 be the maximum degree of G and H, respectively. Then $\lambda(G \boxtimes H) \leq \Delta_1^2\Delta_2^2 + \Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2$.*

Proof. Again, we apply Algorithm 2.1.1 to obtain an $L(2,1)$-labeling with the maximum label k on the graph $G \boxtimes H$. Let $x = (u, v)$ in $V(G) \times V(H)$. Then $\deg_{G\boxtimes H}(x) = \deg_G(u) + \deg_H(v) + \deg_G(u)\deg_H(v)$. Denote $d = \deg_{G\boxtimes H}(x)$, $d_1 = \deg_G(u)$, $d_2 = \deg_H(v)$, $\Delta_1 = \Delta(G)$ and $\Delta_2 = \Delta(H)$. Hence $d = d_1 + d_2 + d_1 d_2$ and $\Delta = \Delta(G \boxtimes H) = \Delta_1 + \Delta_2 + \Delta_1\Delta_2$.

Let G and H be two graphs of order ν_1 and ν_2, respectively. Suppose $V(G) = \{u_1, \dots, u_{\nu_1}\}$ and $V(H) = \{v_1, \dots, v_{\nu_2}\}$. Consider the strong product graph $G \boxtimes H$. We list the vertex set $V(G) \times V(H)$ in lexicographic order. Then the adjacency matrix of $G \boxtimes H$ with respect to this list is $A = A_1 \otimes A_2 + A_1 \otimes I_2 + I_1 \otimes A_2$, where A_1 and A_2 are adjacency matrices of G and H respectively, J_2 is the square matrix of order ν_2 all of whose entries are equal to 1 and I_1 is the identity matrix of order ν_1. Note that $P \otimes Q$ is the Kronecker product of the matrices P and Q.

Then $A^2 + A = (A_1 \otimes A_2)^2 + (A_1 \otimes I_2 + I_1 \otimes A_2)^2 + A_1 \otimes A_2(A_1 \otimes I_2 + I_1 \otimes A_2) + (A_1 \otimes I_2 + I_1 \otimes A_2)A_1 \otimes A_2 + A_1 \otimes A_2 + A_1 \otimes I_2 + I_1 \otimes A_2 = A_1^2 \otimes A_2^2 + A_1^2 \otimes I_2 + 2A_1 \otimes A_2 + I_1 \otimes A_2^2 + 2A_1^2 \otimes A_2 + 2A_1 \otimes A_2^2 + A_1 \otimes A_2 + A_1 \otimes I_2 + I_1 \otimes A_2$

For fixed vertex (u_i, v_j) in $G \boxtimes H$, the number of nonzero entries in the (u_i, v_j)th row of $A^2 + A$ excluding the diagonal entries is the same as the number of nonzero entries in the (u_i, v_j)th row of $A_1^2 \otimes A_2^2 + A_1^2 \otimes I_2 + 2A_1 \otimes A_2 + I_1 \otimes A_2^2 + 2A_1^2 \otimes A_2 + 2A_1 \otimes A_2^2 + A_1 \otimes A_2 + A_1 \otimes I_2 + I_1 \otimes A_2$ excluding the diagonal entries.

Let Δ_1 and Δ_2 be the maximum degrees of G and H, respectively. For fixed vertex (u_i, v_j) in $G \boxtimes H$, we only look at the (u_i, v_j)th row of the above matrix. Then the number of nonzero entries in this row excluding the diagonal entries is at most

$\deg_G(u_i)(\Delta_1 - 1)\deg_H(v_j)(\Delta_2 - 1) + \deg_G(u_i)(\Delta_1 - 1) + \deg_G(u_i)\deg_H(v_j) + \deg_H(v_j)(\Delta_2 - 1) + \deg_G(u_i)(\Delta_1 - 1)\deg_H(v_j) + \deg_G(u_i)\deg_H(v_j)(\Delta_2 - 1) + \deg_G(u_i)\deg_H(v_j) + \deg_G(u_i) + \deg_H(v_j)$.

Thus,

$|I_2| + |I_1| \leq \Delta_1(\Delta_1 - 1)\Delta_2(\Delta_2 - 1) + \Delta_1(\Delta_1 - 1) + \Delta_1\Delta_2 + \Delta_2(\Delta_2 - 1) + \Delta_1(\Delta_1 - 1)\Delta_2 + \Delta_1\Delta_2(\Delta_2 - 1) + \Delta_1\Delta_2 + \Delta_1 + \Delta_2 = \Delta_1^2\Delta_2^2 + \Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2$. ■

In [49] it was proved that $\lambda(G \boxtimes H) \leq \Delta^2 + \Delta - (\Delta_1 + \Delta_2 + 4)\Delta_1\Delta_2\Delta^2 + \Delta_1 + \Delta_2 - 5\Delta_1\Delta_2$.

Because $\Delta^2 + \Delta - (\Delta_1 + \Delta_2 + 4)\Delta_1\Delta_2 - (\Delta_1^2\Delta_2^2 + \Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2) = (\Delta_1 + \Delta_2 - 2)\Delta_1\Delta_2 + \Delta_1 + \Delta_2$, we reduce the bound in [49] by $(\Delta_1 + \Delta_2 - 2)\Delta_1\Delta_2 + \Delta_1 + \Delta_2$.

Chapter 3

$L(2,1)$-Labelings of the Composition of n Graphs

Graph products play an important role in network applications. In [53] the Cartesian product and the composition of two graphs were studied and it was proven that the L(2,1)-labeling number of these graphs is bounded above by the square of the maximum degree (with minor exceptions); unfortunately, the proof for the bound on the L(2,1)-labeling number of the composition of graphs had a mistake, so the bound is only valid for graphs with no isolated vertices. In this chapter we address the problem with the proof in [53] and study the $L(2,1)$-labeling number of the composition of n graphs. We show that the $L(2,1)$-labelling of the composition of n graphs is much smaller than the square of the maximum degree. As corollaries, our bound for the $L(2,1)$-labeling number of the composition of n graphs is better than that given in [60] for the composition of two graphs $G_1[G_2]$ if $\nu_2 < \Delta_2^2 + 1$, where ν_2 and Δ_2 are the number of vertices and maximum degree of G_2 respectively.

3.1 The Combinatorial Analysis Approach

The definition of the composition of two graphs G and H has been provided in Section 1.4.

By the definition of $G[H]$, if $\Delta(G) = 0$, then $G[H]$ consists of disjoint copies of H. Thus $\lambda(G[H]) = \lambda(H)$. Therefore, we assume $\Delta(G) \geq 1$.

The composition of n ($n \geq 2$) graphs $G_1, G_2, ..., G_n$, $C_{G_1,G_2,...,G_n}$, is defined recursively by $C_{G_n} = G_n$ and $C_{G_k,G_{k+1},...,G_n} = G_k[C_{G_{k+1},G_{k+2},...,G_n}]$ for $k = n-1, n-2, ..., 1$.

In this section, we obtain an upper bound for $\lambda(C_{G_1,G_2,...,G_n})$ in terms of the maximum degrees of $G_1, G_2, ..., G_n, C_{G_1,G_2,...,G_n}$.

Theorem 3.1.1 *Let $G_1, G_2, ..., G_n$ be graphs with maximum degrees Δ_1, Δ_2, ..., Δ_n, respectively, such that $\Delta_1 \geq 1$. Then*

$\lambda(C_{G_1,G_2,...,G_n}) \leq \beta_2(1 + \Delta_1 + \Delta_1^2) + \alpha - 1,$

where $\beta_j = |V(G_j)| \times |V(G_{j+1})| \times \cdots \times |V(G_n)|$ for all $j = 1, 2, \ldots, n$, and $\alpha = \sum_{j=2}^{n-1}(\beta_{j+1}\Delta_j) + \Delta_n$.

Proof. Let us apply Algorithm 2.1.1 to $C_{G_1,G_2,...,G_n}$ and let $x = (i_1, i_2, \ldots, i_n) \in V(C_{G_1,G_2,...,G_n})$ be a vertex with the largest label. Let d be the degree of x in

$C_{G_1,G_2,...,G_n}$ and for each $j = 1, 2, \ldots, n$, let us define the following values: d_j is the degree of i_j in G_j, $\nu_j = |V(G_j)|$, and $\beta_j = \nu_j\nu_{j+1}\cdots\nu_n$. Let $\beta_{n+1} = 1$. Note from the definition of composition that the number of vertices of $C_{G_j,...,G_n}$ is β_j, for all $j = 1, 2, \ldots, n$. Let t be the number of vertices at distance 2 from vertex x in graph $C_{G_1,...,G_n}$.

Observe that graph $C_{G_j,...,G_n}, j < n$, can be constructed as follows:

1. Replace each vertex u of G_j with a copy of $C_{G_{j+1},...,G_n}$. Let us denote this copy of $C_{G_{j+1},...,G_n}$ corresponding to vertex u as C^u.

2. For every edge e_{uv} of G_j add an edge between every vertex of C^u and every vertex of C^v.

Therefore, the set of the following vertices contains all the vertices of $C_{G_1,G_2,\ldots,G_n}$ that are at distance two from $x = (i_1, i_2, \ldots, i_n)$:

The vertices in the copy C^{i_1} of $C_{G_2,\ldots,G_n}$ corresponding to vertex i_1, with the exception of x and the neighbours of x in C^{i_1}. The number of vertices in C^{i_1} is $\nu_2\nu_3\cdots\nu_n$ and the number of neighbours of x in C^{i_1} is $d - d_1\nu_2\nu_3\cdots\nu_n$ as d is the total number of neighbours of x and $d_1\nu_2\nu_3\cdots\nu_n$ is the number of neighbours of x that do not belong to C^{i_1}.

There can be at most $d_1(\Delta_1 - 1)\nu_2\nu_3\cdots\nu_n$ vertices not in C^{i_1} at distance 2 from x as each neighbour of i_1 in G_1 has at most Δ_1 neighbors.

Hence,

$$\begin{aligned} t &\le \nu_2\nu_3\cdots\nu_n - (d - d_1\nu_2\cdots\nu_n) - 1 + d_1(\Delta_1 - 1)\nu_2\cdots\nu_n \\ &= \nu_2\cdots\nu_n(1 + d_1\Delta_1 - d_1) - d + d_1\nu_2\cdots\nu_n - 1 \\ &= \beta_2(1 + d_1\Delta_1) - d - 1 \end{aligned}$$

The maximum degree of the graph $C_{G_1,G_2,\ldots,G_n}$ is

$$\begin{aligned} \Delta &= \sum_{j=1}^{n}(\beta_{j+1}\Delta_j) = \beta_2\Delta_1 + \sum_{j=2}^{n}(\beta_{j+1}\Delta_j) \\ &= \beta_2\Delta_1 + \alpha, \text{ where } \alpha = \sum_{j=2}^{n}(\beta_{j+1}\Delta_j). \end{aligned} \tag{3.1}$$

Thus, we obtain the following bound B for the $L(2,1)$-labelling number of $C_{G_1,G_2,\ldots,G_n}$ (see (2.1))

$$\begin{aligned} B &= |I_1| + |I_2| \leq d + d + t \\ &\leq 2d + \beta_2(1 + d_1\Delta_1) - d - 1 \\ &= d + \beta_2(1 + d_1\Delta_1) - 1 \\ &\leq \Delta + \beta_2(1 + d_1\Delta_1) - 1 \\ &= \beta_2\Delta_1 + \alpha + \beta_2(1 + d_1\Delta_1) - 1, \text{ by } (3.1) \\ &= \beta_2(1 + \Delta_1 + d_1\Delta_1) + \alpha - 1. \end{aligned}$$ ■

Corollary 3.1.2 *Let G, H be graphs with maximum degrees Δ_1, Δ_2 respectively, such that $\Delta_1 \geq 1$. Then*

$$\lambda(G[H]) \leq \beta_2(1 + \Delta_1 + \Delta_1^2) + \alpha - 1 = \nu_2\Delta_1 + \Delta_2 - 1 + \nu_2(1 + \Delta_1^2).$$

In [60], Shiu et al. proved that $\lambda(G[H]) \leq \nu_2\Delta_1 + \Delta_2 + \nu_2\Delta_1^2 + \Delta_2^2$. Because $\nu_2\Delta_1 + \Delta_2 + \nu_2\Delta_1^2 + \Delta_2^2 - (\nu_2\Delta_1 + \Delta_2 - 1 + \nu_2(1 + \Delta_1^2)) = \Delta_2^2 - \nu_2 + 1$, the bound in Corollary 3.2.2 is better than that of Shiu et al. if $\nu_2 < \Delta_2^2 + 1$.

Lemma 3.1.3 *Let G_1, G_2 be graphs with maximum degrees Δ_1, Δ_2 and numbers of vertices ν_1, ν_2, respectively, such that $\Delta_1 = 2$ and $\Delta_2 = 0$. Then $\lambda(G_1[G_2]) \leq 5\nu_2 - 1$. In particular, $\lambda(C_5[G_2]) = 5\nu_2 - 1$, where C_5 is a cycle with 5 vertices.*

Proof. Without loss of generality, we can suppose that $G_1 = C_{\nu_1}$, i.e., G_1 is a cycle with $\nu_1(\nu_1 \geq 3)$ vertices. We give an explicit $(5\nu_2 - 1)$-$L(2,1)$-labeling l for $G_1[G_2]$. Let $v_0, ..., v_{\nu_1-2}$ be vertices of C_{ν_1} such that v_i is adjacent to $v_{i+1}, 0 \leq i \leq \nu_1 - 2$ and v_0 is adjacent to v_{ν_1-1}. Then, consider the following cases:

Case 1. $\nu_1 \equiv 0 \bmod 3$.

Subcase 1. $i \equiv 0 \bmod 3$. Label each vertex in each copy of G_2 in $G_1[G_2]$ corresponding to v_i with labels $0, 1, ..., \nu_2 - 1$.

Subcase 2. $i \equiv 1 \bmod 3$. Label each vertex in each copy of G_2 in $G_1[G_2]$ corresponding to v_i with labels $\nu_2+1, \nu_2+2, ..., 2\nu_2$.

Subcase 3. $i \equiv 2 \bmod 3$. Label each vertex in each copy of G_2 in $G_1[G_2]$ corresponding to v_i with labels $2\nu_2+2, 2\nu_2+3, ..., 3\nu_2+1$.

Case 2. $\nu_1 \equiv 1 \bmod 3$. First we label each vertex in the copy of G_2 in $G_1[G_2]$ corresponding to $v_0, ..., v_{\nu_2-2}$ as follows

Subcase 1. $i \equiv 0 \bmod 3$. Label each vertex in the copy of G_2 in $G_1[G_2]$ corresponding to v_i with labels $0, 1, ..., \nu_2 - 1$.

Subcase 2. $i \equiv 1 \bmod 3$. Label each vertex in the copy of G_2 in $G_1[G_2]$ corresponding to v_i with labels $2\nu_2+2, 2\nu_2+3, ..., 3\nu_2+1$.

Subcase 3. $i \equiv 2 \bmod 3$. Label each vertex in the copy of G_2 in $G_1[G_2]$ corresponding to v_i with labels $\nu_2+1, \nu_2+2, ..., 2\nu_2$.

Finally, label each vertex in the copy of G_2 in $G_1[G_2]$ corresponding to v_{ν_2-1} with labels $3\nu_2+2, 3\nu_2+3, ..., 4\nu_2+1$.

Case 3. $\nu_1 \equiv 2 \bmod 3$. First we label each vertex in the copy of G_2 in $G_1[G_2]$ corresponding to vertices $v_0, ..., v_{\nu_2-3}$ as follows

Subcase 1. $i \equiv 0 \bmod 3$. Label each vertex in the copy of G_2 in $G_1[G_2]$ corresponding to v_i with labels $0, 1, ..., \nu_2 - 1$.

Subcase 2. $i \equiv 1 \bmod 3$. First label each vertex except the last one in the copy of G_2 in $G_1[G_2]$ corresponding to v_i with labels $\nu_2+1, \nu_2+2, ..., 2\nu_2-1$ and then label the last vertex with $4\nu_2$.

Subcase 3. $i \equiv 2 \bmod 3$. Label each vertex in the copy of G_2 in $G_1[G_2]$ corresponding to v_i with labels $2\nu_2+1, 2\nu_2+2, ..., 3\nu_2$.

Finally label each vertex in the copy of G_2 in $G_1[G_2]$ corresponding to v_{ν_2-2} as follows: First, label each vertex except the last one in this copy of G_2 with labels $4\nu_2+1, 4\nu_2+2, ..., 5\nu_2-1$ and then label the last vertex with label ν_2. Finally, label each vertex in the copy of G_2 in $G_1[G_2]$ corresponding to v_{ν_2-1} as follows: first label each vertex except the last one in this copy of G_2 with labels $3\nu_2+1, 3\nu_2+2, ..., 4\nu_2-1$

and then label the last vertex with label $2\nu_2$.

It is easy to verify that the above labeling is a valid $L(2,1)$-labeling for $G_1[G_2]$ and, therefore, $\lambda(G_1[G_2]) \leq 5\nu_2 - 1$.

Note that since C_5 is a diameter 2 graph, then $C_5[G_2]$ is also a diameter 2 graph, therefore all vertices of $C_5[G_2]$ must be assigned different labels. Thus, $\lambda(C_5[G_2]) \geq 5\nu_2 - 1$. But since we already showed that $\lambda(C_5[G_2]) \leq 5\nu_2 - 1$, then $\lambda(C_5[G_2]) = 5\nu_2 - 1$. Hence, the above labelling scheme is optimal for $C_5[G_2]$. ■

Lemma 3.1.4 *Let G_1, G_2 be graphs with maximum degrees Δ_1, Δ_2 and numbers of vertices ν_1, ν_2, respectively, such that $\Delta_1 \geq 1$ and $\Delta_2 = 0$. Then $\lambda(C_{G_1,G_2}) \leq \Delta^2 - \Delta$ where Δ is the maximum degree of C_{G_1,G_2}, with the only exceptions that $\lambda(C_{G_1,G_2}) \leq \Delta^2 + \Delta$ when $\Delta_1 \geq 3$ and $\nu_2 = 1$ and $\lambda(C_{G_1,G_2}) = \Delta^2$ when C_{G_1,G_2} consists of copies of C_4.*

Proof. Because $\Delta_2 = 0$, the number of vertices at distance one from x is at most $\nu_2\Delta_1$ and the number of vertices at distance two from x is at most $\nu_2\Delta_1(\Delta_1 - 1) + \nu_2 - 1$. Hence, we can compute the bound B from euqation (2.1) as follows: $|I_1| \leq \nu_2\Delta_1$, $|I_2| \leq \nu_2\Delta_1 + \nu_2\Delta_1(\Delta_1 - 1) + \nu_2 - 1$. Then $B = |I_1| + |I_2| \leq \nu_2\Delta_1 + \nu_2\Delta_1 + \nu_2\Delta_1(\Delta_1 - 1) + \nu_2 - 1 = \nu_2\Delta_1^2 + \nu_2\Delta_1 + \nu_2 - 1$. We need to consider three cases.

Case 1. $\Delta_1 \geq 3$.

Subcase 1. $\nu_2 = 1$. Then $C_{G_1,G_2} = G_1$. In this case, C_{G_1,G_2} is the general graph G_1 with maximum degree $\Delta_1 \geq 3$.

Subcase 2. $\nu_2 \geq 2$. Since $(\nu_2\Delta_1)^2 - \nu_2\Delta_1 - (\nu_2\Delta_1^2 + \nu_2\Delta_1 + \nu_2 - 1) = \nu_2((\nu_2 - 1)\Delta_1^2 - 2\Delta_1 - 1) + 1 \geq \nu_2(9\nu_2 - 16) + 1 = 9\nu_2^2 - 16\nu_2 + 1 \geq 2\nu_2 + 1$. Hence $B \leq (\nu_2\Delta_1)^2 - \nu_2\Delta_1 - (2\nu_2 + 1) = \Delta^2 - \Delta - (2\nu_2 + 1)$.

Case 2. $\Delta_1 = 2$. By Lemma 3.2.3, we have $\lambda(G_1[G_2]) \leq 5\nu_2 - 1$.

But

$(\nu_2\Delta_1)^2 - \nu_2\Delta_1 - (5\nu_2 - 1) = 4\nu_2^2 - 7\nu_2 + 1 \geq 3$, so $\lambda(G_1[G_2]) \leq (\nu_2\Delta_1)^2 - \nu_2\Delta_1 - 3 = \Delta^2 - \Delta - 3$.

Case 3. $\Delta_1 = 1$. Then $\lambda(G_1[G_2]) = 2\nu_2$.

If $\nu_2 \geq 3$, then $(\nu_2\Delta_1)^2 - \nu_2\Delta_1 - 2\nu_2 = \nu_2^2 - 3\nu_2 \geq 0$. Hence $\lambda(G_1[G_2]) \leq \Delta^2 - \Delta$.

If $\nu_2 = 2$, then $G_1[G_2]$ consists of copies of C_4. Hence $\lambda(G_1[G_2]) = 4 \leq \Delta^2$. ■

Lemma 3.1.5 *Let $G_1, G_2, ..., G_n$ be graphs with maximum degrees Δ_1, Δ_2, ..., Δ_n, respectively, such that $\Delta_1 \geq 1$. Then $\lambda(C_{G_1,G_2,...,G_n}) \leq \Delta^2 - \Delta$, where Δ is the maximum degree of $C_{G_1,G_2,...,G_n}$, with the only exceptions that $\lambda(C_{G_1,G_2,...,G_n}) \leq \Delta^2 + \Delta$ when $\nu_2 = \nu_3 = \cdots = \nu_n = 1$ and $\lambda(C_{G_1,G_2,...,G_n}) = \Delta^2$ where $C_{G_1,G_2,...,G_n}$ consists of copies of C_4.*

Proof. From Theorem 3.2.1, $\lambda(C_{G_1,G_2,...,G_n}) \leq \beta_2(1 + \Delta_1 + \Delta_1^2) + \alpha - 1$ so we just need to show that this bound is at most $\Delta^2 - \Delta$, except when $\nu_2 = \nu_3 = \cdots = \nu_n = 1$ or $C_{G_1,G_2,...,G_n}$ consists of copies of C_4. Note that

$$\begin{aligned}
&\Delta^2 - \Delta - (\beta_2(1 + \Delta_1 + \Delta_1^2) + \alpha - 1) \\
&= (\beta_2\Delta_1 + \alpha)^2 - (\beta_2\Delta_1 + \alpha) - (\beta_2(1 + \Delta_1 + \Delta_1^2) + \alpha - 1), \text{ from (3.1)} \\
&= (\beta_2^2 - \beta_2)\Delta_1^2 + 2\beta_2\Delta_1(\alpha - 1) + \alpha^2 - 2\alpha - \beta_2 + 1
\end{aligned}$$

We now need to consider three cases.

Case 1. $\alpha = \sum_{j=2}^{n}(\beta_{j+1}\Delta_j) = 0$. Then $\Delta_j = 0, j = 2, ..., n$. By Lemma 3.2.3, the conclusion holds.

Case 2. $\alpha = \sum_{j=2}^{n}(\beta_{j+1}\Delta_j) = 1$. Then

Subcase 1. $\beta_2 = 1$. Since $\beta_2 = \nu_2\nu_3 \cdots \nu_n = 1$, then $\nu_2 = \nu_3 = \cdots = \nu_n = 1$. Hence $C_{G_1,G_2,...,G_n} = G_1$. In this case, $C_{G_1,G_2,...,G_n}$ is the general graph G_1 with maximum degree $\Delta_1 \geq 1$.

Subcase 2. $\beta_2 \geq 2$. Since $\Delta^2 - \Delta - (\beta_2(1 + \Delta_1 + \Delta_1^2) + \alpha - 1) = (\beta_2^2 - \beta_2)\Delta_1^2 + 2\beta_2\Delta_1(\alpha - 1) + \alpha^2 - 2\alpha - \beta_2 + 1 = (\beta_2^2 - \beta_2)\Delta_1^2 - \beta_2 = \beta_2((\beta_2 - 1)\Delta_1^2 - 1) \geq \beta_2(\Delta_1^2 - 1) \geq 0$. Then the conclusion holds.

Case 3. $\alpha = \sum_{j=2}^{n}(\beta_{j+1}\Delta_j) \geq 2$. Then $\Delta^2 - \Delta - (\beta_2(1 + \Delta_1 + \Delta_1^2) + \alpha - 1) = (\beta_2^2 - \beta_2)\Delta_1^2 + 2\beta_2\Delta_1(\alpha - 1) + \alpha^2 - 2\alpha - \beta_2 + 1 \geq (\beta_2^2 - \beta_2)\Delta_1^2 + \beta_2(2\Delta_1 - 1) + 1 \geq \beta_2^2 + 1($ since $\beta_2 \geq 2$ and $\Delta_1 \geq 1)$. Then the conclusion holds. ■

By the proof of Lemma 3.2.5, the bound in Theorem 3.2.1 is much smaller than $\Delta^2 - \Delta$ if $\alpha \geq 2$ or if $\alpha = 1$ and $\Delta_1 \geq 2$.

3.2 Correction to the Proof in [53] for the Composition of Two Graphs

Theorem 4.3 in [53] states a bound for $\lambda(C_{G_1,G_2})$ by establishing a lower bound on ε, the number of edges of the subgraph F induced by the neighbors of a vertex x labelled with the largest label by algorithm LABEL. Unfortunately, the proof of the theorem given in [53] is not totally correct because if vertex x is isolated in G_2, then the lower bound for ε will not hold and therefore the upper bound for $\lambda(G_1[G_2])$ can not be established by this method but if vertex x is not isolated in G_2, then the lower bound for ε will still hold and therefore the proof is still correct.

In this section, we fix the proof of that Theorem.

Theorem 3.2.1 *[53] Let the maximum degree of $G_1[G_2]$ be Δ. Then $\lambda(G_1[G_2]) \leq \Delta^2 + \Delta - 2\nu_2\Delta_1$ or $\lambda(G_1[G_2]) \leq \Delta^2 - \Delta$, with the only exceptions that $\lambda(C_{G_1,G_2}) \leq \Delta^2 + \Delta$ when $\Delta_1 \geq 3$ and $\nu_2 = 1$ and $\lambda(G_1[G_2]) = \Delta^2$ when $G_1[G_2]$ consists of copies of C_4.*

Proof. We use Algorithm 2.1.1 to obtain an $L(2,1)$-labeling with maximum label k for the graph $G_1[G_2]$. Let $x \in V(G_1[G_2])$ be labeled by k. We only consider the case when the degree of x in G_2 is zero.

Case 1. $\Delta_2 > 0$. Because x is isolated in G_2, the number of vertices at distance one from x is at most $\nu_2\Delta_1$ and the number of vertices at distance two from x is at most $\nu_2\Delta_1(\Delta_1-1)+\nu_2-1$. Hence for computing the bound B from equation (2.1) we get $|I_1| \le \nu_2\Delta_1$, $|I_2| \le \nu_2\Delta_1+\nu_2\Delta_1(\Delta_1-1)+\nu_2-1$. Then $B=|I_1|+|I_2| \le \nu_2\Delta_1+\nu_2\Delta_1+\nu_2\Delta_1(\Delta_1-1)+\nu_2-1 = \nu_2\Delta_1^2+\nu_2\Delta_1+\nu_2-1$.

Since $\Delta_2>0$ and x is isolated in G_2, $\nu_2 \ge 3$. Note that $\Delta_1 \ge 1$ and $\Delta_2 \ge 1$, then $(\nu_2\Delta_1+\Delta_2)^2-(\nu_2\Delta_1+\Delta_2)-(\nu_2\Delta_1^2+\nu_2\Delta_1+\nu_2-1) = \nu_2((\nu_2-1)\Delta_1^2+2\Delta_1(\Delta_2-1))+\Delta_2(\Delta_2-1)+\nu_2-1 \ge \nu_2(\nu_2-1)\Delta_1^2+\nu_2-1 \ge \nu_2(\nu_2-1)+\nu_2-1 = \nu_2^2-1 \ge 8$. Hence $B \le (\nu_2\Delta_1+\Delta_2)^2-(\nu_2\Delta_1+\Delta_2)-(\nu_2^2-1) = \Delta^2-\Delta-(\nu_2^2-1) \le \Delta^2-\Delta-8$.

Case 2. $\Delta_2=0$. The proof is the same as Lemma 3.2.3. ■

Chapter 4

On Some Results for the $L(2,1)$-Labeling on Cartesian Sum Graphs

In [53], [31] and [57], Shao and Yeh, Klavžar and Špacapan, and Shao and Zhang proved that the $L(2,1)$-labeling number of the Cartesian product, the composition, the direct product, the strong product and the Cartesian sum of graphs is bounded by the square of the maximum degree (with minor exceptions). Shao, Klavžar, Shiu and Zhang [49] improved the upper bounds obtained in [31] with a more refined analysis of neighborhoods in product graphs than that used in [31].

In this chapter we consider the Cartesian sum of graphs and derive, both, lower and upper bounds for the $L(2,1)$-labeling number; we use two approaches to derive the upper bounds and both approaches improve previously known bounds. We also present new approximation algorithms for $L(2,1)$-labelings on Cartesian sum graphs.

4.1 Lower and Upper Bounds on the $L(2,1)$-Labelings of Cartesian Sum Graphs

Given a graph G, the number of vertices in G is denoted $\nu(G)$. A vertex u of G is isolated if its degree is zero. The number of isolated vertices in G is denoted $t(G)$. The maximum degree of G is denoted $\Delta(G)$. If u and v are two adjacent vertices of G, the edge connecting them is denoted as uv.

Lemma 4.1.1 *Let G and H be two graphs. Then $G \bigoplus H$ has a subgraph of diameter two with $(\nu(G)-t(G))(\nu(H)-t(H))$ vertices and it also has a subgraph of diameter three with $\max\{\nu(G)(\nu(H)-t(H)), \nu(H)(\nu(G)-t(G))\}$ vertices.*

Proof. Let G' and H' be the subgraphs of G and H obtained by removing all the isolated vertices, respectively. Observe that if G' or H' are empty then the first bound of the Lemma holds trivially, so let us assume that G' and H' are not empty. Let G' and H' consist of connected components $G_1, G_2, ..., G_k$ $(k \geq 1)$ and $H_1, H_2, ..., H_p$ $(p \geq 1)$, respectively. Note that $\nu(G_i) \geq 2$ and $\nu(H_j) \geq 2$ for each connected component $G_i, H_j, i = 1, 2, ..., k, j = 1, 2, ..., p$. Let $(u, v), (u', v')$ be any two nonadjacent vertices of $G \bigoplus H$, where $u \in G_i$, $v' \in H_l$, for $i \in \{1, 2, ..., k\}$ and $l \in \{1, 2, ..., p\}$. Since G_i and H_l are connected, let u'' be a vertex adjacent to u in G_i and let v'' be a vertex adjacent to v' in H_l. By the definition of $G \bigoplus H$, (u, v) and (u'', v'') are adjacent and (u', v') and (u'', v'') are also adjacent. Hence (u, v) and (u', v') are at distance two in $G \bigoplus H$, and so $G \bigoplus H$ has a subgraph $G' \bigoplus H'$ of diameter two with $(\nu(G)-t(G))(\nu(H)-t(H))$ vertices.

We now derive the first part for the second bound of the Lemma. Note that if H' is empty this part of the bound is trivially zero, so we assume that H' is not empty. Let $(u, v), (u', v')$ be two nonadjacent vertices of $G \bigoplus H$, where $u \in G_i, u' \in G_j$ and v, v' are two different vertices in H. Let w, w' be two adjacent vertices in H. Since G_i

and G_j are connected, let u'' be a vertex adjacent to u in G_i and let u''' be a vertex adjacent to u' in G_j. By the definition of $G \bigoplus H$, (u, v) and (u'', w) are adjacent, (u''', w') and (u', v') are adjacent, and (u'', w) and (u''', w') are adjacent. Hence (u, v) and (u', v') are at distance three. Then $G' \bigoplus H$ has a subgraph of diameter three that includes G' and all vertices of H; this subgraph has $\nu(H)(\nu(G) - t(G))$ vertices. Similarly, $G \bigoplus H'$ has a subgraph of diameter three with $\nu(G)(\nu(H) - t(H))$ vertices.
■

Corollary 4.1.2 *Let G and H be two connected graphs. Then $G \bigoplus H$ is of diameter two.*

Theorem 4.1.3 *For any two graphs G and H, $\lambda(G \bigoplus H) \geq (\nu(G) - t(G))$ $(\nu(H) - t(H)) - 1$.*

Proof. By Lemma 4.1.1, $G \bigoplus H$ has a subgraph of diameter two with $(\nu(G) - t(G))(\nu(H) - t(H))$ vertices. Since in an $L(2, 1)$-labeling of a diameter two graph all the vertices must have different labels, then $\lambda(G \bigoplus H) \geq (\nu(G) - t(G))(\nu(H) - t(H)) - 1$. ■

We now compute an upper bound for $\lambda(G \bigoplus H)$.

Theorem 4.1.4 *For any two graphs G and H, $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) - t(G)t(H) + \Delta(G \bigoplus H) - 1$.*

Proof. Note that $G \bigoplus H$ has $t(G)t(H)$ isolated vertices. Thus, the number of vertices within distance two from any vertex x, is at most $\nu(G)\nu(H) - t(G)t(H) - 1$. Therefore, by Algorithm 2.1.1, $\lambda(G \bigoplus H) \leq |I_2| + |I_1| \leq \nu(G)\nu(H) - t(G)t(H) + \Delta(G \bigoplus H) - 1$.
■

In [57] it is proved that $\lambda(G \bigoplus H) \leq D' = (\Delta(G \bigoplus H))^2 - \nu(G)(\Delta(G) - 1)\Delta(H) - \nu(H)(\Delta(H) - 1)\Delta(G) - (\Delta(G) + \Delta(H))\Delta(G)\Delta(H) - \Delta(G) - \Delta(H) + 1$. Let $D =$

$\nu(G)\nu(H) - t(G)t(H) + \Delta(G \bigoplus H) - 1$, be the bound from Theorem 4.1.4. We now compare the bounds D' and D.

Note that $\Delta(G \bigoplus H) = \nu(G)\Delta(H) + \nu(H)\Delta(G) - \Delta(G)\Delta(H) \geq 2(\nu(G)\nu(H)\Delta(G)\Delta(H))^{1/2} - \Delta(G)\Delta(H) = (\Delta(G)\Delta(H))^{1/2}(2(\nu(G)\nu(H))^{1/2} - (\Delta(G)\Delta(H))^{1/2}) = (\Delta(G)\Delta(H))^{1/2}((\nu(G)\nu(H))^{1/2} + (\nu(G)\nu(H))^{1/2} - (\Delta(G)\Delta(H))^{1/2}) \geq (\Delta(G)\Delta(H))^{1/2}((\nu(G)\nu(H))^{1/2} + (\Delta(G)\Delta(H) + \Delta(G) + \Delta(H) + 1)^{1/2} - (\Delta(G)\Delta(H))^{1/2}) > (\Delta(G)\Delta(H))^{1/2}(\nu(G)\nu(H))^{1/2}$, the second inequality follows from $\nu(G) \geq \Delta(G) + 1$ and $\nu(H) \geq \Delta(H) + 1$.

Thus, $(\Delta(G \bigoplus H))^2 > \nu(G)\nu(H)\Delta(G)\Delta(H)$ and so

$D' - D = [(\Delta(G \bigoplus H))^2 - \nu(G)(\Delta(G) - 1)\Delta(H) - \nu(H)(\Delta(H) - 1)\Delta(G) - (\Delta(G) + \Delta(H))\Delta(G)\Delta(H) - \Delta(G) - \Delta(H) + 1] - [\nu(G)\nu(H) - t(G)t(H) + \Delta(G \bigoplus H) - 1] = [(\Delta(G \bigoplus H))^2 - \nu(G)(\Delta(G) - 1)\Delta(H) - \nu(H)(\Delta(H) - 1)\Delta(G) - (\Delta(G) + \Delta(H))\Delta(G)\Delta(H) - \Delta(G) - \Delta(H) + 1] - [\nu(G)\nu(H) - t(G)t(H) + (\nu(G)\Delta(H) + \nu(H)\Delta(G) - \Delta(G)\Delta(H)) - 1] = (\Delta(G \bigoplus H))^2 - (\nu(G)\nu(H) - t(G)t(H)) - (\nu(G)\Delta(G)\Delta(H) + \nu(H)\Delta(H)\Delta(G) + (\Delta(G) + \Delta(H) - 1)\Delta(G)\Delta(H) + \Delta(G) + \Delta(H) - 2) > \nu(G)\nu(H)\Delta(G)\Delta(H) - (\nu(G)\nu(H) - t(G)t(H)) - (\nu(G)\Delta(G)\Delta(H) + \nu(H)\Delta(H)\Delta(G) + (\Delta(G) + \Delta(H) - 1)\Delta(G)\Delta(H) + \Delta(G) + \Delta(H) - 2)$.

Noting again that $\nu(G) \geq \Delta(G) + 1$ and $\nu(H) \geq \Delta(H) + 1$, we conclude that $D' - D = \Theta(\nu(G)\nu(H)\Delta(G)\Delta(H))$. So, our bound is asymptotically better than in [57].

4.2 Algorithm BlockLabel

In this section, we present a different algorithm for computing an $L(2,1)$-labeling for the Cartesian sum of two graphs that is better than the algorithm presented in the Section 4.1.

In the vertex coloring problem the goal is to color the vertices of a given graph G

with the minimum possible number of colors so that adjacent vertices have different colors. The minimum number of colors needed to color the vertices of a graph G is called the chromatic number of G, denoted $\chi(G)$. Consider two graphs G, H and optimum colorings χ_G, χ_H for them. Without loss of generality, let the colors assigned to the vertices of G and H be $1, ..., \chi(G)$ and $1, ..., \chi(H)$ respectively; moreover, let all the isolated vertices in G and H be assigned color 1. We partition the vertices of $G \bigoplus H$ into blocks, as follows. All vertices (u, v) of $G \bigoplus H$ where u has color i and v has color j are placed in block B_{ij}. Let B be the set of all these blocks. We use the following algorithm for labeling $G \bigoplus H$.

Algorithm BlockLabel(B)

Input: Set B of blocks as described above.

Output: The maximum label used in an $L(2, 1)$-labeling for the vertices in B.

1. Sort the blocks in B in any order.
2. $l \leftarrow 0$.
3. <u>For</u> each block $B_{ij} \in B$ <u>do</u> {
4.   <u>If</u> $i = 1$ and $j = 1$ <u>then</u> {
5.   <u>For</u> each vertex $u \in B_{11}$ <u>do</u> {
6.   <u>If</u> u is isolated in $G \bigoplus H$ <u>then</u> Assign u label 0.
7.   <u>otherwise</u> Assign u label l and then set $l \leftarrow l + 1$.

        }

    }

8.   <u>otherwise</u>
9.   <u>For</u> each vertex $u \in B_{ij}$ <u>do</u> Assign u label l and then set $l \leftarrow l + 1$.
10. $l \leftarrow l + 1$//skip a label.

}

11. Return $l - 1$.

Theorem 4.2.1 *Let G and H be two graphs. Then one of the following holds.*

a. If both G and H are not complete graphs or odd cycles, then $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) - t(G)t(H) + \Delta(G)\Delta(H) - 2$;

b. If both G and H are odd cycles, then $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + 7$;

c. If both G and H are complete graphs, then $\lambda(G \bigoplus H) = 2\nu(G)\nu(H) - 2$;

d. If one of G and H is not a complete graph or odd cycle, but the other is an odd cycle, then $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + 3\Delta(G) - 2$ or $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + 3\Delta(H) - 2$;

e. If one of G and H is not a complete graph or odd cycle, but the other is a complete graph, then $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + \Delta(G)\nu(H) - 2$ or $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + \Delta(H)\nu(G) - 2$;

f. If one of G and H is a complete graph and the other is an odd cycle, then $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + 3\nu(G) - 2$ or $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + 3\nu(H) - 2$.

Proof. We first show that algorithm BlockLabel produces an $L(2,1)$-labeling for $G \bigoplus H$. Let us consider the non-isolated vertices in some block $B_{ij} \in B$. For any two vertices (u, v) and (u', v') in B_{ij}, since u and u' have the same color i in χ_G, then u and u' are at distance at least two in G; similarly, v and v' are at distance at least two in H. By the definition of $G \bigoplus H$, (u, v) and (u', v') are at distance at least two in $G \bigoplus H$. Thus, all the vertices in block $B_{i,j}$ can be labelled consecutively.

Now let us consider the vertices in two different blocks. For any two vertices (u, v) and (u', v') from two different blocks of B, there is the possibility that they are adjacent in $G \bigoplus H$. Note that since in algorithm BlockLabel at least one label has been skipped between the labelling of (u, v) and (u', v'), then the labels of these vertices differ by at least 2 and so the above labelling scheme is feasible.

The number of labels used to label $G \bigoplus H$ is equal to the number $\nu(G)\nu(H) - t(G)t(H)$ of non-isolated vertices plus the number of labels skipped in step 10 of the algorithm. Notice that the number of labels skipped is equal to the number of blocks in B minus 1.

Since the number of blocks in B is at most $\chi(G)\chi(H)$, then $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) -$

$t(G)t(H) + \chi(G)\chi(H) - 2$. We can now combine this result with Brooks Theorem to prove (a)- (e).

(a). If both G and H are not complete graphs or odd cycles, then $\chi(G) \leq \Delta(G)$ and $\chi(H) \leq \Delta(H)$. And the conclusion follows.

(b). If both G and H are odd cycles, then $t(G) = t(H) = 0$ and $\chi(G) = \chi(H) = 3$. Thus, $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) - t(G)t(H) + \chi(G)\chi(H) - 2 \leq \nu(G)\nu(H) + 7$.

(c). If both G and H are complete graphs , then $G \bigoplus H$ is a complete graph. Thus, $\lambda(G \bigoplus H) = 2\nu(G)\nu(H) - 2$.

(d). If G is not a complete graph or odd cycle, and H is an odd cycle, then $t(H) = 0$, $\chi(G) \leq \Delta(G)$ and $\chi(H) = 3$. So, $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) - t(G)t(H) + \chi(G)\chi(H) - 2 \leq \nu(G)\nu(H) + 3\Delta(G) - 2$. The other case is similar.

(e). If G is not a complete graph or odd cycle, and H is a complete graph, then $t(H) = 0$, $\chi(H) = \nu(H)$ and $\chi(G) \leq \Delta(G)$. So, $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) - t(G)t(H) + \chi(G)\chi(H) - 2 \leq \nu(G)\nu(H) + \Delta(G)\nu(H) - 2$. The other case is similar.

(f). If G is a complete graph and H is an odd cycle, then $t(G) = t(H) = 0$, $\chi(G) = \nu(G)$ and $\chi(H) = 3$. So, $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + 3\nu(G) - 2$. The other case is similar. ∎

We now compare the bounds in Theorem 4.1.4 and Theorem 4.2.1. Note that $[\nu(G)\nu(H) - t(G)t(H) + \Delta(G \bigoplus H) - 1] - [\nu(G)\nu(H) - t(G)t(H) + \chi(G)\chi(H) - 2] \geq \Delta(G)\nu(H) + \Delta(H)\nu(G) - \Delta(G)\Delta(H) - (\Delta(G)+1)(\Delta(H)+1) = \Delta(G)(\nu(H) - \Delta(H)) + \Delta(H)(\nu(G) - \Delta(G)) - \Delta(G) - \Delta(H) - 1 = \Delta(G)(\nu(H) - \Delta(H) - 1) + \Delta(H)(\nu(G) - \Delta(G) - 1) - 1 \geq \Delta(G) + \Delta(H) - 1$. Then, the bound in Theorem 4.2.1 is better than that in Theorem 4.1.4.

The following Corollaries follow from Theorem 4.2.1.

Corollary 4.2.2 *Let $G \bigoplus H$ be the Cartesian sum of any two graphs G and H and let both G and H have non-isolated vertices. Then one of the following holds.*
a. If both G and H are not complete graphs or odd cycles, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio $(\nu(G)\nu(H) - t(G)t(H) + \Delta(G)\Delta(H) -$

$2)/((\nu(G) - t(G))(\nu(H) - t(H)) - 1)$;

b. If both G and H are odd cycles, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio $(\nu(G)\nu(H) + 7)/((\nu(G) - t(G))(\nu(H) - t(H)) - 1)$;

c. If both G and H are complete graphs, then there is an exact algorithm to $L(2,1)$-label $G \bigoplus H$ with $\lambda(G \bigoplus H) = 2\nu(G)\nu(H) - 2$;

d. If one of G and H is not a complete graph or odd cycle, but the other is an odd cycle, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio $(\nu(G)\nu(H) + 3\Delta(G) - 2)/((\nu(G) - t(G))(\nu(H) - t(H)) - 1)$ or $(\nu(G)\nu(H) + 3\Delta(H) - 2)/((\nu(G) - t(G))(\nu(H) - t(H)) - 1)$;

e. If one of G and H is not a complete graph or odd cycle, but the other is a complete graph, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio $(\nu(G)\nu(H) + \Delta(G)\nu(H) - 2)/((\nu(G) - t(G))(\nu(H) - t(H)) - 1)$ or $(\nu(G)\nu(H) + \Delta(H)\nu(G) - 2)/((\nu(G) - t(G))(\nu(H) - t(H)) - 1)$;

f. If one of G and H is a complete graph and the other is an odd cycle, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio $(\nu(G)\nu(H) + 3\nu(G) - 2)/((\nu(G) - t(G))(\nu(H) - t(H)) - 1)$ or $(\nu(G)\nu(H) + 3\nu(H) - 2)/((\nu(G) - t(G))(\nu(H) - t(H)) - 1)$.

Corollary 4.2.3 *Let $G \bigoplus H$ be the Cartesian sum of two connected graphs G and H. Then one of the following holds.*

a. If both G and H are not complete graphs or odd cycles, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio less than 2;

b. If both G and H are odd cycles, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio $(\nu(G)\nu(H) + 7)/((\nu(G)\nu(H) - 1)$;

c. If both G and H are complete graphs, then there is an exact algorithm to $L(2,1)$-label $G \bigoplus H$ with $\lambda(G \bigoplus H) = 2\nu(G)\nu(H) - 2$;

d. If one of G and H is not a complete graph or odd cycle, but the other is an odd cycle, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio $(\nu(G)\nu(H)+3\Delta(G)-2)/((\nu(G)\nu(H)-1)$ or $(\nu(G)\nu(H)+3\Delta(H)-2)/((\nu(G)\nu(H)-1)$;

e. If one of G and H is not a complete graph or odd cycle, but the other is a complete graph, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio $(\nu(G)\nu(H) + \Delta(G)\nu(H) - 2)/((\nu(G)\nu(H) - 1)$ or $(\nu(G)\nu(H) + \Delta(H)\nu(G) - 2)/((\nu(G)\nu(H) - 1)$;

f. If one of G and H is a complete graph and the other is an odd cycle, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio $(\nu(G)\nu(H) + 3\nu(G) - 2)/((\nu(G)\nu(H) - 1)$ or $(\nu(G)\nu(H) + 3\nu(H) - 2)/((\nu(G)\nu(H) - 1)$.

Chapter 5

$L(2,1)$-Labelings of Disk Graphs

Roberts [45] and Sakai [46] pointed out that the class of unit interval graphs and its generalization, the class of unit d-disk graphs, are of particular interest in the frequency assignment problem. When transmitters are located in R^d, for $d = 1, 2$ or 3, interference takes place if two transmitters are within a certain distance from each other, so interfering transmitters can be conveniently represented with a unit d-disk graph. Unfortunately, no useful characterizations of unit d-disk graphs is known except for $d = 1$. Sakai [46] gave upper bounds for the $L(2,1)$-labeling number of unit 1-disk graphs. Recently, Shao et al. [55] characterized unit d-disk graphs for $d = 2, 3$ and gave upper bounds for the $L(2,1)$-labeling number for this class of graphs.

In this chapter, we characterize d-disk graphs for $d > 1$, and give the first upper bounds on the $L(2,1)$-labeling number for this classes of graphs.

5.1 $L(2,1)$-Labelings of d-Disk Graphs

Shao et al. [55] characterized unit d-disk graphs for $d = 2, 3$ and proved that $\lambda(G) \leq \frac{4}{5}\Delta^2 + 2\Delta$ for any unit 2-disk graph G and $\lambda(G) \leq \frac{11}{12}\Delta^2 + 2\Delta$ for any unit 3-disk graph G.

In this section, we characterize d-disk graphs for $d \geq 2$ and give the first upper bounds for the $L(2,1)-$labeling number for this class of graphs.

From the definition of d-disk graphs, we observe that the set of all d-disk graphs is $K_{1,n}$-free if and only if it is not possible to pack n d-spheres around and touching a central d-sphere without the surrounding spheres touching each other. Hence, we consider the problem of bounding the minimum value n such that for any d-disk graph G, G is $K_{1,n}$-free, as this will allow us to bound the $L(2,1)$-labeling number for d-disk graphs.

We first show that the smallest value n for which any 2-disk graph of diameter ratio σ is $K_{1,n}$-free is $n = \lceil \pi / \arcsin(1/(\sigma+1)) \rceil$.

Theorem 5.1.1 *Every 2-disk graph of diameter ratio σ is $K_{1,n}$-free for every $n \geq \lceil \pi / \arcsin(1/(\sigma+1)) \rceil$.*

Proof. A 2-disk graph of diameter ratio σ is $K_{1,n}$-free if and only if for any collection D' of 2-spheres or circles of diameter ratio σ it is not possible to pack n circles from D' around and touching a central circle $C_0 \in D'$, without the surrounding circles touching each other.

Let $n(\sigma)$ be the smallest value such that every 2-disk graph of diameter ratio σ is $K_{1,n(\sigma)}$-free. Let $D = \{C_0, C_1, \ldots, C_{n(\sigma)-1}\}$ be a collection of $n(\sigma)$ circles of diameter ratio σ such that $C_1, \ldots, C_{n(\sigma)-1}$ can be placed around C_0 in such a way that each $C_i \neq C_0$ touches C_0 and no two $C_i, C_j \neq C_0$ touch each other. Let d_{min}, d_{max} be the minimum and maximum diameters of the circles in D, respectively. Note that by the definition of $n(\sigma)$, D has the property that no additional circle of diameter $d, d_{min} \leq d \leq d_{max}$, can be packed around C_0 along with $C_1, \ldots, C_{n(\sigma)-1}$ without causing the circles surrounding C_0 to touch each other.

Without loss of generality, let C_0 have radius 1. Consider a packing of $C_1, \ldots, C_{n(\sigma)-1}$ around C_0 as described above. Let us assume that each circle $C_i, i > 0$, is glued to C_0 at the point P_i where they touch. Note that if we reduce the radius of some circle C_i while keeping C_i glued to C_0 at P_i, the distance between C_i and its two adjacent circles does not decrease (see Figure 5.1), so no intersections among circles can be created by this operation.

If we reduce the radius of each circle $C_j, 1 \leq j \leq n(\sigma) - 1$, to $1/\sigma$, we get a new packing where circles $C_i, 1 \leq i \leq n(\sigma) - 1$ do not touch and the corresponding disk graph has diameter ratio σ. In this new packing consider two adjacent circles C_i, C_j (see Figure 5.2). The angle θ between the centers of C_i, C_0, and C_j is $\theta > 2\arcsin(1/(\sigma+1))$ and thus $n(\sigma) \leq \lceil 2\pi/\theta \rceil \leq \lceil \pi/\arcsin(1/(\sigma+1)) \rceil = n'$.

It is now easy to show that every disk graph of diameter ratio σ is $K_{1,n}$-free for $n \geq n'$. To see this, for the sake of contradiction, let G be a disk graph of diameter ratio σ and let G have $K_{1,n}$ as an induced subgraph. Let $D_{1,n}$ be the disk representation of $K_{1,n}$. This means that $D_{1,n}$ consists of n circles $C_1, C_2, \ldots, C_n$ that can be packed around a central circle $C_0 \in D_{1,n}$, but as shown above, this is impossible for $n \geq n'$. ■

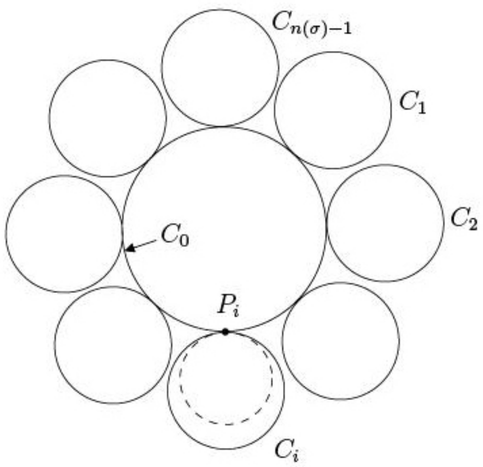

Fig. 5.1: Reducing the radius of a circle C_i does not decrease the distance from C_i to its neighbouring circles.

We now turn our attention to d-disk graphs for $d \geq 3$ and compute an upper

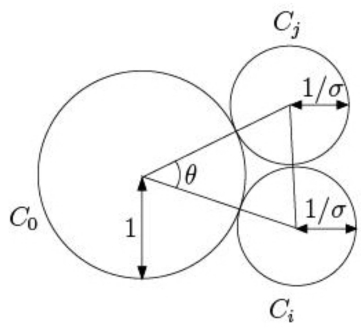

Fig. 5.2: Angle θ between two neighbouring circles

bound on the minimum value n such that any d-disk graph is $K_{1,n}$-free.

Consider a set D of d-spheres of diameter ratio σ with $|D| = d+1$. Let the d-spheres in D be tangent to each other. Let d of these spheres have radius $1/\sigma$ and the remaining one, S_0, have radius 1. The centers of the spheres in D delimit a d-simplex * $\triangle_d$ with *edge* lengths $2/\sigma$ and $1/\sigma + 1$. Let the center of S_0 be v. Consider a new d-sphere S with center v and radius $1/\sigma + 1$ (See Figure 5.3). Observe that all the centers of the spheres in D, except S_0, are on S and, obviously, $\triangle_d \subset S$. Consider the d faces of $\triangle_d$ that intersect at the center v of S. Let us extend these faces away from v until they intersect S. The region C delimited by these extended faces and the section of S above them is called a *spherical sector*. An example of $\triangle_3$ and the corresponding spherical sector in 3 dimensions is shown in Figure 5.3.

Let the volume of S be $V(S)$ and the volume of C be $V(C)$. Let $n_\sigma = \lceil \frac{V(S)}{V(C)} \rceil$.

Lemma 5.1.2 *Every d-disk graph of diameter ratio σ is K_{1,n_σ}-free, for $d \geq 3$.*

Proof. Consider a d-disk graph G of diameter ratio σ and m vertices. Let $D = \{C_1, \ldots, C_m\}$ be a disk representation for G. Let $C_i \in D$ be a d-sphere for which

*A *d-simplex* is a polytope of dimension d with $d+1$ vertices (cf. [68]).

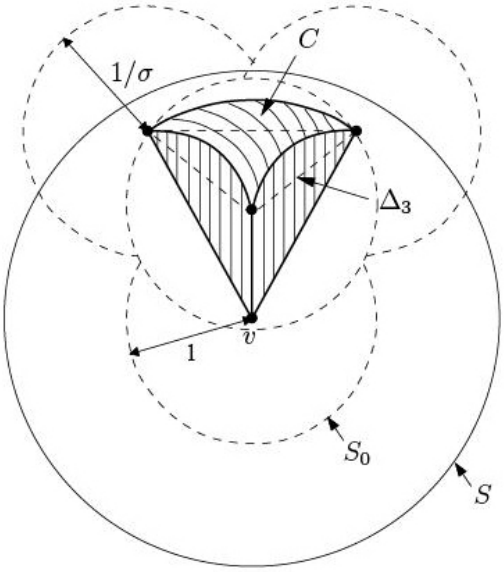

Fig. 5.3: Spherical sector in 3 dimensions.

spheres $C_{i_1}, ..., C_{i_{k_i}}$ are packed around and touching C_i, but spheres C_{i_j} do not touch each other. Proceeding as in the proof of Theorem 5.1.1 we can convert D into a new disk representation for G in which C_i has radius 1 and each sphere $C_{i_j}, j = 1, 2, ..., k_i$, has radius $1/\sigma$. Then, by the way in which the spherical sector C and sphere S have been defined, we note that $k_i < \lceil \frac{V(S)}{V(C)} \rceil = n_\sigma$ as it is not possible to pack n_σ spheres around C_i as described above, so G must be K_{1,n_σ}-free. ∎

We cite the following theorem by Shao et al. [55].

Theorem 5.1.3 *If G is $K_{1,n}$-free then $\lambda(G) \leq \frac{n-2}{n-1}\Delta^2 + 2\Delta$, where Δ is the maximum degree of G.*

By Theorem 5.2.1, Lemma 5.2.2 and Theorem 5.2.3, we have the following Theorem.

Theorem 5.1.4 *Let G be a d-disk graph of diameter ratio σ and maximum degree*

Δ, *for* $d \geq 2$. *Then*

$$\lambda(G) \leq \begin{cases} \frac{\lceil \pi/\arcsin(1/(\sigma+1)) \rceil - 2}{\lceil \pi/\arcsin(1/(\sigma+1)) \rceil - 1}\Delta^2 + 2\Delta & \text{if } d = 2, \\ \frac{n_\sigma - 2}{n_\sigma - 1}\Delta^2 + 2\Delta & \text{if } d \geq 3. \end{cases}$$

We now compute the exact values of n_σ for $d = 3$ and then give an estimation of n_σ for $d > 3$.

Lemma 5.1.5 $n_\sigma = \lceil (4\pi)/(3\arccos((\sigma^2 + 2\sigma - 1)/(2\sigma^2 + 4\sigma)) - \pi) \rceil$ *if* $d = 3$.

Proof. First let us recall the definition of a *spherical triangle* [1]: a spherical triangle consists of three vertices on the surface of a sphere S and three sides which are the arcs of the short segments of great circles that join pairs of these vertices. Note that, for a spherical sector C in 3 dimensions, as defined above (see Figure 5.3), its non-planar face is delimited by a spherical triangle.

Consider 3 spheres S_1, S_2, S_3 of radius $1/\sigma$ and a sphere S_0 of radius 1 with center v, tangent to each other as shown in Figure 5.3. Let S be a sphere of radius $1+1/\sigma$ and center v and let T be the spherical triangle delimiting the spherical sector C defined by these 4 spheres. Let the three angles of T, the area of S and the area of T be α,β,γ, $A(S)$ and $A(T)$, respectively. By Girards formula, $A(T) = R^2(\alpha + \beta + \gamma - \pi)$ [1], where $R = 1 + 1/\sigma$ is the radius of S. Since $\cos(\alpha) = \cos(\beta) = \cos(\gamma) = (\sigma^2 + 2\sigma - 1)/(2\sigma^2 + 4\sigma)$, then $A(T) = R^2(3\arccos((\sigma^2 + 2\sigma - 1)/(2\sigma^2 + 4\sigma)) - \pi)$. Thus, $n_\sigma = \lceil \frac{V(d)}{V(C)} \rceil = \lceil \frac{A(S)}{A(T)} \rceil = \lceil (4\pi R^2)/(R^2(3\arccos((\sigma^2+2\sigma-1)/(2\sigma^2+4\sigma)) - \pi)) \rceil = \lceil (4\pi)/(3\arccos((\sigma^2 + 2\sigma - 1)/(2\sigma^2 + 4\sigma)) - \pi) \rceil$. ∎

Lemma 5.1.6 $n_\sigma < \lceil d!(1/\sigma + 1)^{d-1}\pi^{d/2}/(\Gamma(d/2+1)d^{1/2}(2/\sigma)^{(d-1)/2} \rceil$ *(where* $\Gamma(x)$ *is the gamma function) if* $d \geq 4$.

Proof. Consider a set D of d-spheres tangent to each other in which $|D| = d + 1$ and where d of the spheres have radius $1/\sigma$ and a central d-sphere S_0 has radius 1. Let the center of S_0 be v. The centers of the spheres are the vertices of a d-simplex $\triangle_d$. Consider also a sphere S of center v and radius $1/\sigma + 1$. Let us draw

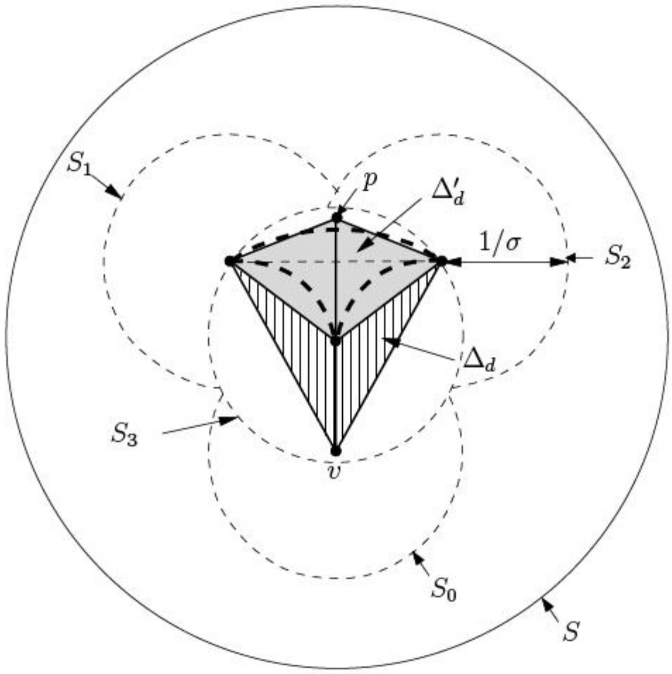

Fig. 5.4: d-Simplexes $\triangle_d$ and $\triangle'_d$.

a straight line l from v perpendicular* to the regular $(d-1)$-simplex $\triangle_{d-1}$ formed by all the vertices of $\triangle_d$ except v, and let the intersection point of l and S be p. By connecting p with all the vertices of $\triangle_d$ except v, we get another d-simplex $\triangle'_d$ which shares a common $(d-1)$-simplex $\triangle_{d-1}$ with $\triangle_d$ (See Figure 5.4). Because $\triangle_d$ and $\triangle'_d$ share a common $(d-1)$-simplex $\triangle_{d-1}$ and both v and p are on a line perpendicular to the regular $(d-1)$-simplex $\triangle_{d-1}$, the combined volume of $\triangle_d$ and $\triangle'_d$ is $RV(\triangle_{d-1})/d$, where R is the radius of the d-sphere S and $V(\triangle_{d-1})$ is the volume of the regular $(d-1)$-simplex $\triangle_{d-1}$. Since the volume of a d-sphere of radius R is $V(R) = \pi^{d/2}R^d/\Gamma(d/2+1)$ [42] and the volume of a regular d-simplex $\triangle'$ with edge length q is $V(\triangle') = (d+1)^{1/2}q^d/((d)!2^{d/2})$ [11], then, the volume of sphere S is $V(S) = \pi^{d/2}(1/\sigma+$

*A line is perpendicular to the simplex if it is orthogonal to each face of the simplex.

$1)^d/\Gamma(d/2+1)$ and $V(\triangle_d)+V(\triangle'_d) = (1/d)(1/\sigma+1)d^{1/2}(2/\sigma)^{d-1}/((d-1)!2^{(d-1)/2})$ (as the volume of the regular $(d-1)$-simplex $\triangle_{d-1}$ is $d^{1/2}(2/\sigma)^{d-1}/((d-1)!2^{(d-1)/2})$). Thus, $n_\sigma = \lceil\frac{V(S)}{V(C)}\rceil < \lceil\frac{V(S)}{V(\triangle_d)+V(\triangle'_d)}\rceil = \lceil(\pi^{d/2}(1/\sigma+1)^d/\Gamma(d/2+1))/((1/d)(1/\sigma+1)d^{1/2}(2/\sigma)^{d-1}/((d-1)!2^{(d-1)/2}))\rceil = \lceil d!(1/\sigma+1)^{d-1}\pi^{d/2}/(\Gamma(d/2+1)d^{1/2}(2/\sigma)^{(d-1)/2}\rceil$.

We note that we can also upper bound n_σ by $\frac{A(S)}{V(\triangle_{d-1})}$, where $A(S)$ is the area of S. Since the area of a d-sphere of radius R is $dV(R)/R = d\pi^{d/2}R^d/(\Gamma(d/2+1)R)$ [42], then, $A(S) = d\pi^{d/2}(1/\sigma+1)^{d-1}/\Gamma(d/2+1)$. So, $n_\sigma < \lceil\frac{A(S)}{V(\triangle_{d-1})}\rceil = \lceil(d\pi^{d/2}(1/\sigma+1)^{d-1}/(\Gamma(d/2+1))/(d^{1/2}(2/\sigma)^{d-1}/((d-1)!2^{(d-1)/2}))\rceil = \lceil d!(1/\sigma+1)^{d-1}\pi^{d/2}/(\Gamma(d/2+1)d^{1/2}(2/\sigma)^{(d-1)/2}\rceil$. Surprisingly, the two approaches yield the same upper bound for n_σ. ■

Note that $\Gamma(d/2+1) = \begin{cases} (d/2)! & \text{if } d \text{ is even,} \\ (\pi)^{1/2}((d+1)!/((d+1)/2)!)2^{-(d+1)} & \text{if } d \text{ is odd.} \end{cases}$

Thus, $n_\sigma(d) < \begin{cases} \lceil d!/(d/2)!(\pi/d)^{1/2}((\sigma+1)^2\pi/(2\sigma))^{1/2})^{(d-1)/2}\rceil & \text{if } d \text{ is even,} \\ \lceil((d+1)/2)!(4/(d+1))d^{-1/2}(2(\sigma+1)^2\pi/\sigma)^{1/2})^{(d-1)/2}\rceil & \text{if } d \text{ is odd.} \end{cases}$

Chapter 6

$L(2,1)$-Labelings of Total Graphs

Given an undirected graph $G = (V, E)$ the total graph $T(G)$ of G has one vertex for each vertex and edge of G and it has an edge between u and v if either u and v are adjacent or incident in G. The class of total graphs is a generalization of the class of line or edge graphs [48]. In [56], Shao et al. derived the first known upper bounds for the $L(2,1)$-labeling number of total graphs. In this chapter, we compute upper bounds for the $L(2,1)$-labeling number of total graphs of $K_{1,n}$-free graphs, where $K_{1,n}$ is the complete bipartite graph with one vertex in one side of the partition and n in the other.

6.1 Total Graphs

A *total coloring* of a graph $G = (V, E)$ is a coloring of $V(G) \cup E(G)$ such that no two adjacent or incident edges or vertices receive the same color. The *total chromatic number* $\chi_t(G)$ of G is the smallest number of colors in a total coloring of G. If G is simple it is not hard to see that $\chi_t(G) = \chi(T(G))$. It is known that $\chi_t(G) \geq \Delta(G)+1$ for G simple and a well-known conjecture says that $\chi_t(G) \leq \Delta(G) + 2$ (cf. [26]).

A *total* $L(2,1)$*-labeling* of a graph $G = (V, E)$ is a function f from $V \cup E$ to the set of all nonnegative integers such that

•$|f(x) - f(y)| \geq 2$ if one of the following conditions hold: (i) $x, y \in V$ and $(x, y) \in E$, or (ii) $(x, y) \in E$, and x and y are incident on the same vertex, or (iii) $x \in V$ and $y \in E$ and y is incident on x.

•$|f(x) - f(y)| \geq 1$ if one of the following conditions hold: (i) $x, y \in V$ and $(x, z), (z, y) \in E$, or (ii) x=$(t, z), y = (z', p) \in E$,$t \neq p$ and $(z, z') \in E$ or (iii) $x \in V$ and $y = (t, t') \in E$ and x is is adjacent to t or t'.

The total $L(2,1)$-labeling number $\lambda_t(G)$ of G is the smallest number k such that G has a total $L(2,1)-$labeling with $\max\{f(v) : v \in V \cup E\} = k$. If G is simple, then it is not hard to see that $\lambda_t(G) = \lambda(T(G))$.

6.2 Total $L(2,1)$-Labelings of $K_{1,n}$-Free Graphs

A $K_{1,n}$-*free graph* G is a graph that contains no induced subgraph $K_{1,n}$. Note that $\Delta \geq n$. A $K_{1,3}-$free graph is also called a claw-free graph. Claw-free graphs have been extensively studied due to their interesting properties and large number of applications [13]. A $K_{1,2}$-free graph G is a complete graph K_n, and the exact value of $\lambda(T(K_n))$ has been computed in [56]. Therefore, in this paper we assume $n \geq 3$.

Let $G = (V, E)$ be a graph, the complement $\overline{G} = (V, \overline{E})$ of G is the graph with the same vertex set as G but $(u, v) \in \overline{E}$ if and only if $(u, v) \notin E$. Let K_n be a complete graph with n vertices; its complement $\overline{K_n}$ is a graph with n vertices and no edges. Let the largest number of edges of any graph H spanning p vertices and not containing an induced forbidding subgraph F, be denoted as $ex(p, F)$.

To upper bound the total labeling number of a $K_{1,n}$-free graph we will need the following result from Turán.

Theorem 6.2.1 *[64] For all $p \geq n, ex(p, K_n) = \frac{(n-2)(p^2-r^2)}{2(n-1)} + \frac{r(r-1)}{2}$, where $p \equiv r$ $\pmod{n-1}$ and $0 \leq r < n-1$.*

Let Δ be the maximum degree of G. If $\Delta = 1$, then G consists of disjoint copies of K_1 and $T(G)$ consists of disjoint copies of K_3, so $\lambda(T(G)) = 4$ in this case. If $\Delta = 2$, then G consists of a collection of paths and/or cycles and so G contains $K_{1,2}$. Since as mentioned above we consider only the case when $n \geq 3$ in the sequel we assume that $\Delta > 2$.

Let the number of edges of a graph G be denoted as $\varepsilon(G)$.

Theorem 6.2.2 *If G is $K_{1,n}$-free then $\lambda(T(G)) \leq \frac{2n-3}{4n-4}\Delta_T^2 + \frac{3n-1}{2n-2}\Delta_T - 1$, where Δ_T is the maximum degree of $T(G)$.*

Proof. Let x be a vertex of $T(G)$. To prove the theorem we will use inequality (2.1), which requires that we bound the maximum number of vertices at distance 1 and 2 from x in $T(G)$. To do this we consider 2 cases.

Case 1: x is a v-vertex of $T(G)$. Let p and p_T be the degrees of x in G and $T(G)$, respectively. As we observed above, $p_T = 2p$ and $\Delta_T = 2\Delta$, where Δ is the maximum degree of G. We now determine the number of vertices at distance 2 from x. The subgraph H induced in $T(G)$ by all vertices adjacent to x is composed of a complete graph C with p e-vertices, p_1 edges, a graph K with p v-vertices, and p edges connecting vertices of C with vertices of K (see Figure 6.1). Note that since G is $K_{1,n}$-free the graph K does not contain $\overline{K_n}$ as a subgraph and, hence, $\overline{K}$ does not contain K_n. Therefore, by Theorem 6.2.1, $\varepsilon(\overline{K}) \leq \frac{(n-2)(p^2-r^2)}{2(n-1)} + \frac{r(r-1)}{2}$, and so

$$\varepsilon(K) \geq \frac{p(p-1)}{2} - \frac{(n-2)(p^2-r^2)}{2(n-1)} - \frac{r(r-1)}{2}$$

Let $f(p)$ be the number of vertices at distance 2 from x in $T(G)$. From Figure 6.2 observe that all vertices at distance 2 from x must be neighbours of vertices in K; this is because the neighbours of the e-vertices in C at distance 2 from x are also neighbours of vertices in K. Since there are p vertices in K and the maximum degree of $T(G)$ is Δ_T, then the maximum number of vertices at distance 2 from x can be

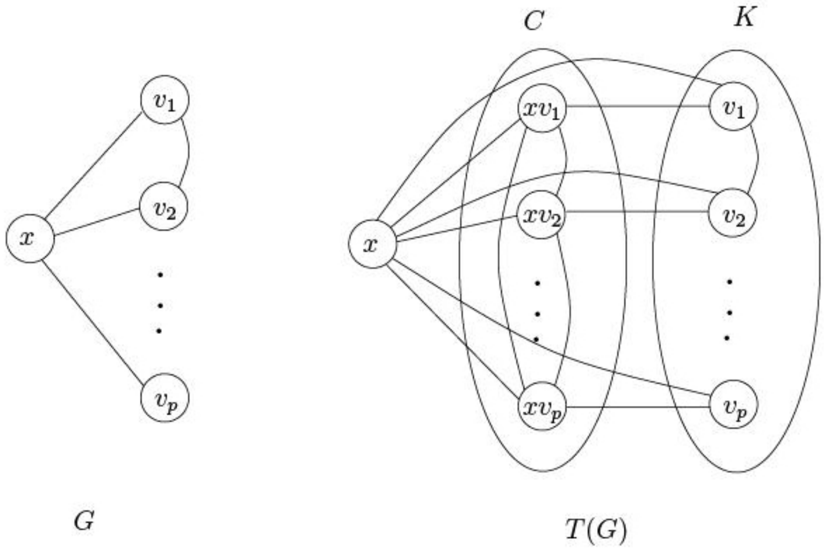

Fig. 6.1: Subgraph induced in $T(G)$ by vertex x and its neighbours in G.

at most $p(\Delta_T - 1)$. To this number we need to subtract $2\varepsilon(K)$ (as the edges in K connect neighbours of x) and $2p$ (as there are p edges connecting vertices in C and K). Therefore,

$$\begin{aligned} f(p) &\le p(\Delta_T - 1) - 2\varepsilon(K) - 2p \\ &= p(\Delta_T - 1) - 2(\tfrac{p(p-1)}{2} - \tfrac{(n-2)(p^2-r^2)}{2(n-1)} - \tfrac{r(r-1)}{2} + p) \\ &= p(2\Delta - 1) - 2(\tfrac{p(p+1)}{2} - \tfrac{(n-2)(p^2-r^2)}{2(n-1)} - \tfrac{r(r-1)}{2}) \\ &= p(2\Delta - 2) - \tfrac{1}{n-1}p^2 + \tfrac{1}{n-1}r^2 - r \end{aligned}$$

Let $g(r) = \frac{1}{(n-1)}r^2 - r$, since $0 \le r < n-1$, the maximum of $g(r)$ happens at $r = 0$ with $g(0) = 0$. Hence $g(r) \le 0$ and so

$$f(p_1) \le 2p\Delta - \frac{1}{n-1}p^2 - 2p$$

Let $h(p) = 2p\Delta - \frac{1}{n-1}p^2 - 2p$, for $0 \le p \le \Delta$, $n \ge 3$. Note that the maximum of $h(p)$ happens at $p = (n-1)(\Delta-1)$. However, since $p \leq \Delta$ and $n \leq 3$, then $2(\Delta-1) \leq p = (n-1)(\Delta-1) \leq \Delta$, which implies that $\Delta \leq 2$. Since, as mentioned above, we only consider the case when $\Delta > 2$, then the maximum value that $h(p)$ can have is $\Delta^2(2-\frac{1}{n-1}) - 2\Delta = \frac{2n-3}{n-1}\Delta^2 - 2\Delta = \frac{2n-3}{4n-4}\Delta_T^2 - \Delta_T$ when $p = \Delta$. Hence $f(p_1) \leq h(p_1) \leq \frac{2n-3}{4n-4}\Delta_T^2 - \Delta_T$.

Case 2: x is an e-vertex of $T(G)$. Let (v, v') be the edge corresponding to x in G. Let $\deg_G v = p$ and $\deg_G v' = p'$, so there are $p + p'$ neighbors of x in $T(G)$.

To determine the number of vertices at distance 2 from x, let us first consider vertex v. Let H_v be the subgraph induced in $T(G)$ by all vertices adjacent to v in $T(G)$, except v' and the e-vertices adjacent to the neighbors of v in G, except v'. (see Figure 6.2). H_v is composed of a complete subgraph C, a subgraph K with $p-1$ v-vertices, and a subgraph L formed by e-vertices not adjacent to v. Since G is $K_{1,n}$-free then K does not contain $\overline{K_n}$ and thus $\overline{K}$ does not contain K_n. By Theorem 6.2.1, $ex(p-1, K_n) = \frac{(n-2)((p-1)^2-r^2)}{2(n-1)} + \frac{r(r-1)}{2}$, where $p-1 \equiv r(\bmod{(n-1)})$ and $0 \leq r < n-1$, hence, $\varepsilon(\overline{K}) \leq \frac{(n-2)((p-1)^2-r^2)}{2(n-1)} + \frac{r(r-1)}{2}$, and

$$\varepsilon(K) \geq \frac{(p-1)(p-2)}{2} - \frac{(n-2)((p-1)^2-r^2)}{2(n-1)} - \frac{r(r-1)}{2}$$

From Figure 6.2 observe that the vertices at distance 2 from x are the v-vertices in K plus the e-vertices in L; let the number of these latter e-vertices be $f(p)$. Since there are $p-1$ vertices in K and the maximum degree of G is Δ, then the maximum number of e-vertices in L is at most $(p-1)(\Delta-1)$; to this number we need to subtract $\varepsilon(K)$, as otherwise the e-vertices adjacent to two v-vertices in K would be counted twice (see Figure 6.2). Therefore

$$\begin{aligned} f(p) &\leq (p-1)(\Delta-1) - (\tfrac{(p-1)(p-2)}{2} - \tfrac{(n-2)((p-1)^2-r^2)}{2(n-1)} - \tfrac{r(r-1)}{2}) \\ &= (p-1)(\Delta - p/2) + \tfrac{n-2}{2(n-1)}(p-1)^2 + \tfrac{1}{2(n-1)}r^2 - r/2 \end{aligned}$$

Since $0 \leq r < n-1$, then $\frac{1}{2(n-1)}r^2 - r/2 \leq 0$, so

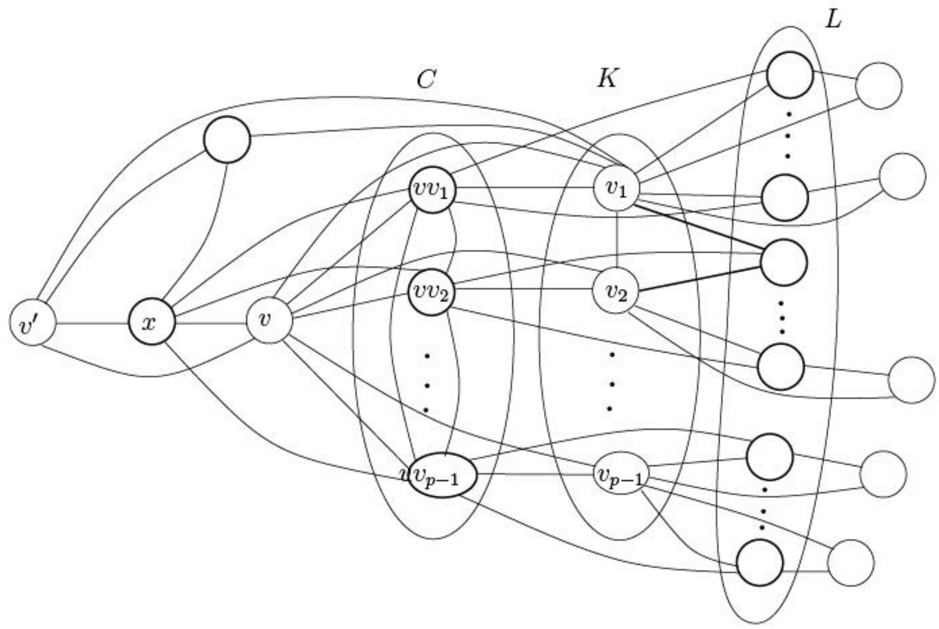

Fig. 6.2:

$$f(p) \leq (p-1)(\Delta - p/2) + \frac{n-2}{2(n-1)}(p-1)^2$$

Let $h(p) = (p-1)(\Delta - p/2) + \frac{n-2}{2(n-1)}(p-1)^2$, for $0 \leq p \leq \Delta$, $n \geq 3$. The maximum value of $h(p)$ is $(\Delta-1)\Delta/2 + \frac{n-2}{2(n-1)}(\Delta-1)^2$, achieved when $p = \Delta$. Hence $f(p) \leq (\Delta-1)\Delta/2 + \frac{n-2}{2(n-1)}(\Delta-1)^2 = \frac{2n-3}{2n-2}\Delta^2 - \frac{3n-5}{2n-2}\Delta + \frac{n-2}{2n-2}$.

Therefore, the number of vertices at distance 2 from x in H_v is at most $p - 1 + \frac{2n-3}{2n-2}\Delta^2 - \frac{3n-5}{2n-2}\Delta + \frac{n-2}{2n-2}$.

We now consider the vertex v' and the subgraph $H_{v'}$, defined similarly as H_v; proceeding as above we determine that the maximum number of vertices at distance 2 from x in $H_{v'}$ is at most $p' - 1 + \frac{2n-3}{2n-2}\Delta^2 - \frac{3n-5}{2n-2}\Delta + \frac{n-2}{2n-2}$.

Hence, the number of vertices at distance 2 from x is at most $(p-1) + (p'-1) + 2(\frac{2n-3}{2n-2}\Delta^2 - \frac{3n-5}{2n-2}\Delta + \frac{n-2}{2n-2}) \leq (\Delta-1) + (\Delta-1) + 2(\frac{2n-3}{2n-2}\Delta^2 - \frac{3n-5}{2n-2}\Delta + \frac{n-2}{2n-2}) = \frac{2n-3}{n-1}\Delta^2 - \frac{n-3}{n-1}\Delta - \frac{n}{n-1} = \frac{2n-3}{4n-4}\Delta_T^2 - \frac{n-3}{2n-2}\Delta_T - \frac{n}{n-1}$.

Combining Case 1 and Case 2, the maximum number of vertices at distance 2 from a vertex x of $T(G)$ is at most $\Delta_T + \max\{\frac{2n-3}{4n-4}\Delta_T^2 - \Delta_T, \frac{2n-3}{4n-4}\Delta_T^2 - \frac{n-3}{2n-2}\Delta_T - \frac{n}{n-1}\} = \frac{2n-3}{4n-4}\Delta_T^2 + (1 - \frac{n-3}{2n-2})\Delta_T - \frac{n}{n-1} = \frac{2n-3}{4n-4}\Delta_T^2 + \frac{n+1}{2n-2}\Delta_T - \frac{n}{n-1}$.

Finally, by (2.1), $\lambda(T(G)) \leq |I_2| + |I_1| \leq \frac{2n-3}{4n-4}\Delta_T^2 + \frac{3n-1}{2n-2}\Delta_T - \frac{n}{n-1}$. ■

Corollary 6.2.3 $\lambda(T(G)) \leq \frac{3}{8}\Delta_T^2 + 2\Delta_T - \frac{3}{2}$ *for any claw-free graph* G.

Given a graph G, the *line graph* of G, $L(G)$, has as set of vertices the edges of G, and two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are incident on the same vertex. A graph G is *linear* if $G = L(H)$ for some graph H. It is easy to see that a linear graph is claw-free. Hence,

Corollary 6.2.4 $\lambda(G) \leq \frac{3}{8}\Delta^2 + 2\Delta - \frac{3}{2}$ *for any linear graph* G *with maximum degree* Δ.

Chapter 7

More results on $L(2,1)$-Labelings of Product Graphs

Graph products play an important role in connecting many useful networks and a great deal of research has been done regarding the $L(2,1)$-labelings on graph products. The Cartesian product, the lexicographic product, the direct product and the strong product form the four standard graph products [25]. In [53] and [31], it was proved that the $L(2,1)$-labeling numbers of the four standard product graphs are bounded by the square of their maximum degrees (with minor exceptions). Recently, Shiu, Shao, Poon and Zhang [60] presented an approach based on the analysis of the adjacency matrices of graphs to derive upper bounds for the $L(2,1)$-labeling numbers of the product graphs. By using this approach, they achieved significant improvements upon previous bounds. In [50], the composition of n graphs was considered. In this chapter, we study the graphs formed by the four standard products of graphs and get significant improvements on their L(2,1)-labeling numbers over previous best known results.

7.1 The Cartesian Product of Graphs

In [53] and [60], upper bounds on $\lambda(G\Box H)$ in terms of the maximum degree $\Delta(G\Box H)$ of $G\Box H$ for any two graphs G and H were obtained. In this section, we get some new results.

It is known that given a vertex u of a graph G, the number of non-zero entries in the u-th row of the adjacency matrix A of G is equal to the number of neighbours of u in G. Similarly, the number of non-zero entries, excluding the diagonal entries, in the u-th row of the matrix A^2 is the number of vertices at distance 2 from u in G and the number of non-zero entries, excluding the diagonal entries, in the u-th row of the matrix $A^2 + A$ is the number of vertices at distance at most 2 from u in G. Hence, to bound the maximum label used by the Algorithm 2.1.1 we count the number of non-zero entries, excluding the diagonal ones, in the matrices A and $A^2 + A$, where A is the adjacency matrix of the input graph. Given two graphs G and H, in the sequel ν_1 and ν_2 denote the number of vertices in G and H, respectively, and Δ_1 and Δ_2 denote the maximum degrees of G and H, respectively.

Theorem 7.1.1 *Let Δ_1 and Δ_2 be maximum degrees of G and H, respectively. Let ν_1 and ν_2 be the numbers of vertices of G and H, respectively. Then* $\lambda(G\Box H) \leq \min\{\nu_1 + \Delta_2^2 + \Delta_1\Delta_2, \nu_2 + \Delta_1^2 + \Delta_1\Delta_2, \nu_1 + \nu_2 + \Delta_1\Delta_2 - 1\} + \Delta_1 + \Delta_2 - 1.$

Proof. We use the Algorithm 2.1.1 to find an $L(2,1)$-labeling of $G\Box H$. Let $x = (u, v)$ in $V(G) \times V(H)$ be a vertex with the largest label k. Note that $deg_{G\Box H}(x) = deg_G(u) + deg_H(v)$. Denote $d = deg_{G\Box H}(x)$, $d_1 = deg_G(u)$, $d_2 = deg_H(v)$, $\Delta_1 = \Delta(G)$ and $\Delta_2 = \Delta(H)$. Hence, $d = d_1 + d_2$ and $\Delta = \Delta(G\Box H) = \Delta_1 + \Delta_2$.

Let the number of vertices of G and H be ν_1 and ν_2, respectively. The adjacency matrix of $G\Box H$ can be expressed as $A = A_1 \otimes I_2 + I_1 \otimes A_2$, where A_1 and A_2 are adjacency matrices of G and H, respectively, I_1 and I_2 are the identity matrices of order ν_1 and ν_2, respectively. $P \otimes Q$ is the Kronecker product of the matrices P and Q.

By [60], for fixed vertex (u_i, v_j) in $G\Box H$, the number of nonzero entries in the (u_i, v_j)th row of $A^2 + A$ excluding the diagonal entries is the same as the number of nonzero entries in the (u_i, v_j)th row of $A_1^2 \otimes I_2 + A_1 \otimes A_2 + I_1 \otimes A_2^2 + A_1 \otimes I_2 + I_1 \otimes A_2$ excluding the diagonal entries.

Observe that the number of nonzero entries in this row excluding the diagonal entries is number of vertices at distance at most 2 from x; this number is at most the minimum of the following three (which means that each one may be better than any other under different conditions):

i) $\deg_G(u_i)(\Delta_1 - 1) + \deg_G(u_i)\deg_H(v_j) + (\nu_2 - 1 - \deg_H(v_j)) + \deg_G(u_i) + \deg_H(v_j) = \deg_G(u_i)\Delta_1 + \nu_2 - 1 + \deg_G(u_i)\deg_H(v_j)$.

ii) $(\nu_1 - 1 - \deg_G(u_i)) + \deg_G(u_i)\deg_H(v_j) + \deg_H(v_j)(\Delta_2 - 1) + \deg_G(u_i) + \deg_H(v_j) = \nu_1 - 1 + \deg_H(v_j)\Delta_2 + \deg_G(u_i)\deg_H(v_j)$.

iii) $(\nu_1 - 1 - \deg_G(u_i)) + \deg_G(u_i)\deg_H(v_j) + (\nu_2 - 1 - \deg_H(v_j)) + \deg_G(u_i) + \deg_H(v_j) = \nu_1 + \nu_2 + \deg_G(u_i)\deg_H(v_j) - 2$.

Thus, $\lambda(G\Box H) \leq |I_2| + |I_1| \leq \min\{\nu_1 + \Delta_2^2 + \Delta_1\Delta_2, \nu_2 + \Delta_1^2 + \Delta_1\Delta_2, \nu_1 + \nu_2 + \Delta_1\Delta_2 - 1\} + \Delta_1 + \Delta_2 - 1$. ■

In [53] and [60], it was proven that $\lambda(G\Box H) \leq \Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2 + \Delta_1 + \Delta_2$. In the following lemma, we prove that the three new bounds are better than the results in [53] and [60] in some cases.

Corollary 7.1.2 *The bounds in Theorem 7.1.1 are better than those in [53] and [60] if $\Delta_1^2 \geq \nu_1 - 1$, or $\Delta_2^2 \geq \nu_2 - 1$ or $\Delta_1^2 + \Delta_2^2 \geq \nu_1\nu_2 - 2$.*

Proof. Since $\Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2 + \Delta_1 + \Delta_2 - (\nu_1 + \Delta_2^2 + \Delta_1\Delta_2 + \Delta_1 + \Delta_2 - 1) = \Delta_1^2 - \nu_1 + 1$, the first bound in Theorem 7.1.1 is better than that in [53] and [60] if $\Delta_1^2 \geq \nu_1 - 1$.

Since $\Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2 + \Delta_1 + \Delta_2 - (\nu_2 + \Delta_1^2 + \Delta_1\Delta_2 + \Delta_1 + \Delta_2 - 1) = \Delta_2^2 - \nu_2 + 1$, our bound is better if $\Delta_2^2 \geq \nu_2 - 1$.

Since $\Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2 + \Delta_1 + \Delta_2 - (\nu_1 + \nu_2 + \Delta_1\Delta_2 + \Delta_1 + \Delta_2 - 2) = \Delta_1^2 + \Delta_2^2 - \nu_1 - \nu_2 + 2$, our bound is better if $\Delta_1^2 + \Delta_2^2 \geq \nu_1\nu_2 - 2$. ■

7.2 The Composition of Graphs

In [53] and [60], upper bounds for $\lambda(G[H])$ in terms of the maximum degree $\Delta(G[H])$ of $G[H]$ for any two graphs G and H were obtained. In this section, we get some new results.

Theorem 7.2.1 *Let Δ_1 and Δ_2 be maximum degrees of G and H, respectively. Let ν_1 and ν_2 be the numbers of vertices of G and H, respectively. Then* $\lambda(G[H]) \leq \min\{(\nu_1 - 1)\nu_2 + \Delta_2^2, \Delta_1^2\nu_2 + \nu_2 - 1, \nu_1\nu_2 - 1\} + \Delta_2 + \Delta_1\nu_2$.

Proof. We use the Algorithm 2.1.1 to find an $L(2,1)$-labeling of $G[H]$. Let $x = (u, v) \in V(G) \times V(H)(= V(G[H]))$ be a vertex with the largest label k. Note that $d_1 = deg_G(u)$, $d_2 = deg_H(v)$, $\Delta_1 = \Delta(G)$, $\Delta_2 = \Delta(H)$ and $n = |V(H)|$. Then $d = deg_{G[H]}(x) = nd_1 + d_2$ and hence, $\Delta = n\Delta_1 + \Delta_2$.

Let the number of vertices of G and H be ν_1 and ν_2, respectively. The adjacency matrix of $G[H]$ can be expressed as $A = A_1 \otimes J_2 + I_1 \otimes A_2$, where A_1 and A_2 are adjacency matrices of G and H respectively, J_2 is the square matrix of order ν_2 all of whose entries are equal to 1 and I_1 is the identity matrix of order ν_1.

By [60], for fixed vertex (u_i, v_j) in $G[H]$, the number of nonzero entries in the (u_i, v_j)th row of $A^2 + A$ excluding the diagonal entries is the same as the number of nonzero entries in the (u_i, v_j)th row of $A_1^2 \otimes J_2 + A_1 \otimes J_2 + I_1 \otimes A_2^2 + I_1 \otimes A_2$ excluding the diagonal entries. Observe that the number of nonzero entries in this row excluding the diagonal entries is number of vertices at distance at most 2 from x; this number is at most the minimum of the following three:

i) $(\nu_1 - 1 - \deg_G(u_i))\nu_2 + \deg_G(u_i)\nu_2 + \deg_H(v_j)(\Delta_2 - 1) + \deg_H(v_j) = (\nu_1 - 1)\nu_2 + \deg_H(v_j)\Delta_2$.

ii) $\deg_G(u_i)(\Delta_1 - 1)\nu_2 + \deg_G(u_i)\nu_2 + (\nu_2 - 1 - \deg_H(v_j)) + \deg_H(v_j) = \deg_G(u_i)\Delta_1\nu_2 + (\nu_2 - 1)$.

iii) $(\nu_1 - 1 - \deg_G(u_i))\nu_2 + \deg_G(u_i)\nu_2 + (\nu_2 - 1 - \deg_H(v_j)) + \deg_H(v_j) = \nu_1\nu_2 - 1$.

Thus, $|I_2| + |I_1| \leq \min\{(\nu_1 - 1)\nu_2 + \Delta_2^2, \Delta_1^2\nu_2 + \nu_2 - 1, \nu_1\nu_2 - 1\} + \Delta_2 + \Delta_1\nu_2$. ■

Upper bounds for $\lambda(G[H])$ are given in [60] and [50]. In the following two lemmas, we prove that the above three new bounds are better than those in [60] and [50] in some cases. In [60] it was proven that $\lambda(G[H]) \leq \Delta_1^2\nu_2 + \Delta_2^2 - 1 + \Delta_1\nu_2 + \Delta_2$.

Corollary 7.2.2 *The bounds in Theorem 7.2.1 are better than those in [60] if* $(\Delta_1^2 - \nu_1 + 1)\nu_2 > 1$, *or* $\Delta_2^2 > \nu_2$ *or* $(\Delta_1^2 - \nu_1)\nu_2 + \Delta_2^2 > 0$.

Proof. Since $(\Delta_1^2\nu_2 + \Delta_2^2 - 1 + \Delta_1\nu_2 + \Delta_2) - ((\nu_1 - 1)\nu_2 + \Delta_2^2 + \Delta_2 + \Delta_1\nu_2) = (\Delta_1^2 - \nu_1 + 1)\nu_2 - 1$, the first bound in Theorem 7.2.1 is better than that in [60] if $(\Delta_1^2 - \nu_1 + 1)\nu_2 > 1$.

Since $(\Delta_1^2\nu_2 + \Delta_2^2 - 1 + \Delta_1\nu_2 + \Delta_2) - (\Delta_1^2\nu_2 + (\nu_2 - 1) + \Delta_2 + \Delta_1\nu_2) = \Delta_2^2 - 1 - (\nu_2 - 1) = \Delta_2^2 - \nu_2$, our bound is better if $\Delta_2^2 > \nu_2$.

Since $(\Delta_1^2\nu_2 + \Delta_2^2 - 1 + \Delta_1\nu_2 + \Delta_2) - (\nu_1\nu_2 + \Delta_2 + \Delta_1\nu_2 - 1) = (\Delta_1^2 - \nu_1)\nu_2 + \Delta_2^2$, our bound is better if $(\Delta_1^2 - \nu_1)\nu_2 + \Delta_2^2 > 0$. ■

In [50] it was proven that $\lambda(G[H]) \leq \nu_2\Delta_1 + \Delta_2 - 1 + \nu_2(1 + \Delta_1^2)$ if $\Delta_1 \geq 1$.

Corollary 7.2.3 *The bounds in Theorem 7.2.1 are better than those in [50] if* $\nu_2(\Delta_1^2 - \nu_1 + 2) > 1 + \Delta_2^2$, *or* $\Delta_1^2 \geq \nu_1$.

Proof. Since $(\nu_2\Delta_1 + \Delta_2 - 1 + \nu_2(1 + \Delta_1^2)) - ((\nu_1 - 1)\nu_2 + \Delta_2^2 + \nu_2\Delta_1 + \Delta_2) = \nu_2(\Delta_1^2 - \nu_1 + 2) - 1 - \Delta_2^2$, the first bound in Theorem 7.2.1 is better than that in [50] if $\nu_2(\Delta_1^2 - \nu_1 + 2) > 1 + \Delta_2^2$.

Since $(\nu_2\Delta_1 + \Delta_2 - 1 + \nu_2(1 + \Delta_1^2)) - (\Delta_1^2\nu_2 + (\nu_2 - 1) + \Delta_2 + \Delta_1\nu_2) = 0$, the second bound is the same as that in [50], except that there is a restriction $\Delta_1 \geq 1$ in [50].

Since $(\nu_2\Delta_1 + \Delta_2 - 1 + \nu_2(1 + \Delta_1^2)) - (\nu_1\nu_2 + \Delta_2 + \Delta_1\nu_2 - 1) = \nu_2(1 + \Delta_1^2 - \nu_1)$, our bound is better if $\Delta_1^2 \geq \nu_1$. ■

The composition of n ($n \geq 2$) graphs $G_1, G_2, ..., G_n$, $C_{G_1,G_2,...,G_n}$, is defined recursively by $C_{G_n} = G_n$ and $C_{G_k,G_{k+1},...,G_n} = G_k[C_{G_{k+1},G_{k+2},...,G_n}]$ for $k = n-1, n-2, ..., 1$.

Corollary 7.2.4 *Let $G_1, G_2, ..., G_n$ be graphs with maximum degrees Δ_1, Δ_2, ..., Δ_n, respectively. Then* $\lambda(C_{G_1,G_2,...,G_n}) \leq \min\{(\nu_1-1)\beta_2+\alpha^2+\alpha+\Delta_1\beta_2, \beta_2(1+\Delta_1+\Delta_1^2)+\alpha-1, \nu_1\beta_2+\alpha+\Delta_1\beta_2-1\}$*, where* $\beta_j = |V(G_j)| \times |V(G_{j+1})| \times \cdots \times |V(G_n)|$ *for all* $j = 1, 2, \ldots, n$*, and* $\alpha = \sum_{j=2}^{n-1}(\beta_{j+1}\Delta_j)+\Delta_n$.

Proof. Let $\Delta = \Delta(C_{G_1,G_2,...,G_n}), \nu = \nu(C_{G_1,G_2,...,G_n}) = \beta_1$, $D_1 = \Delta(G_1)$ and $D_2 = \Delta[C_{G_2,G_3,...,G_n}], \nu_2 = \nu(C_{G_2,G_3,...,G_n}) = \beta_2$, then

$\Delta = \nu_2 D_1 + D_2 = \sum_{j=1}^{n}(\beta_{j+1}\Delta_j) = \beta_2\Delta_1 + \sum_{j=2}^{n}(\beta_{j+1}\Delta_j) = \beta_2\Delta_1+\alpha, D_1 = \Delta_1$ and $D_2 = \alpha$, where $\alpha = \sum_{j=2}^{n}(\beta_{j+1}\Delta_j)$,

Since $C_{G_1,G_2,...,G_n} = G_1[C_{G_2,G_3,...,G_n}]$, then by Theorem 7.2.1,

$(\nu_1-1)\nu_2 + D_2^2 + D_2 + D_1\nu_2 = (\nu_1-1)\beta_2+\alpha^2+\alpha+\Delta_1\beta_2.$

$\nu_2 D_1 + D_2 - 1 + \nu_2(1+D_1^2) = \beta_2\Delta_1+\alpha-1+\beta_2(1+\Delta_1^2) = \beta_2(1+\Delta_1+\Delta_1^2)+\alpha-1.$

$\nu_1\nu_2 + D_2 + D_1\nu_2 - 1 = \nu_1\beta_2+\alpha+\Delta_1\beta_2-1.$ ■

Note that we do not need the restriction $\Delta_1 \geq 1$, unlike Theorem 4.1 of [50] and they only proved $\lambda(C_{G_1,G_2,...,G_n}) \leq \beta_2(1+\Delta_1+\Delta_1^2)+\alpha-1$.

7.3 The Direct Product of Graphs

In [60], upper bounds on $\lambda(G \times H)$ in terms of the maximum degree $\Delta(G \times H)$ of $G \times H$ for any two graphs G and H were obtained. In this section, we get some new results.

Theorem 7.3.1 *Let Δ_1 and Δ_2 be maximum degrees of G and H, respectively. Let ν_1 and ν_2 be the numbers of vertices of G and H, respectively. Then* $\lambda(G \times H) \leq \min\{\Delta_1^2\nu_2 - \Delta_1^2\Delta_2 - \Delta_1^2 - \Delta_1\nu_2 + \Delta_1\Delta_2 + \Delta_1, \Delta_2^2\nu_1 - \Delta_2^2\Delta_1 - \Delta_2^2 - \Delta_2\nu_1 + \Delta_1\Delta_2 + \Delta_2, \nu_1\nu_2 - \nu_1\Delta_2 - \Delta_1\nu_2 + \Delta_1 + \Delta_2 - \nu_1 - \nu_2 + 1\} + 3\Delta_1\Delta_2$.

Proof. We use the Algorithm 2.1.1 to find an $L(2,1)$-labeling of $G \times H$. Let $x = (u, v) \in V(G) \times V(H)$ be a vertex with the largest label k. Note that $deg_{G\times H}(x) =$

$deg_G(u)deg_H(v)$. Then $d = deg_{G\times H}(x)$, $d_1 = deg_G(u)$, $d_2 = deg_H(v)$, $\Delta_1 = \Delta(G)$ and $\Delta_2 = \Delta(H)$. Hence, $d = d_1d_2$ and $\Delta = \Delta(G\times H) = \Delta_1\Delta_2$.

Let the number of vertices of G and H be ν_1 and ν_2, respectively. The adjacency matrix of $G\times H$ can be expressed as $A = A_1 \otimes A_2$, where A_1 and A_2 are adjacency matrices of G and H respectively.

By [60], for fixed vertex (u_i, v_j) in $G\times H$, the number of nonzero entries in the (u_i, v_j)th row of $A^2 + A$ excluding the diagonal entries is the same as the number of nonzero entries in the (u_i, v_j)th row of $A_1^2 \otimes A_2^2 + A_1 \otimes A_2$ excluding the diagonal entries. Observe that the number of nonzero entries in this row excluding the diagonal entries is number of vertices at distance at most 2 from x; this number is at most the minimum of the following three:

i) $\deg_G(u_i)(\Delta_1 - 1)(\nu_2 - 1 - \deg_H(v_j)) + \deg_G(u_i)\deg_H(v_j) \leq \Delta_1(\Delta_1 - 1)(\nu_2 - 1) + \Delta_1\Delta_2 = (\Delta_1^2 - \Delta_1)(\nu_2 - 1) + \Delta_1\Delta_2 = \Delta_1^2\nu_2 - \Delta_1^2 - \Delta_1\nu_2 + \Delta_1\Delta_2 + \Delta_1$.

ii) $(\nu_1 - 1 - \deg_G(u_i))\deg_H(v_j)(\Delta_2 - 1) + \deg_G(u_i)\deg_H(v_j) \leq (\nu_1 - 1)\Delta_2(\Delta_2 - 1) + \Delta_1\Delta_2 = (\nu_1 - 1)(\Delta_2^2 - \Delta_2) + \Delta_1\Delta_2 = \Delta_2^2\nu_1 - \Delta_2^2 - \Delta_2\nu_1 + \Delta_1\Delta_2 + \Delta_2$.

iii) $(\nu_1 - 1 - \deg_G(u_i))(\nu_2 - 1 - \deg_H(v_j)) + \deg_G(u_i)\deg_H(v_j) \leq (\nu_1 - 1)(\nu_2 - 1) + \Delta_1\Delta_2 = (\nu_1\nu_2 - \nu_1 - \nu_1 + 1) + \Delta_1\Delta_2$.

Thus, $|I_2| + |I_1| \leq \min\{\Delta_1^2\nu_2 - \Delta_1^2 - \Delta_1\nu_2 + \Delta_1\Delta_2 + \Delta_1, \Delta_2^2\nu_1 - \Delta_2^2 - \Delta_2\nu_1 + \Delta_1\Delta_2 + \Delta_2, (\nu_1\nu_2 - \nu_1 - \nu_1 + 1) + \Delta_1\Delta_2\} + \Delta_1\Delta_2$. ∎

In [60], it was proven that $\lambda(G\times H) \leq \Delta_1^2\Delta_2^2 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 + 3\Delta_1\Delta_2$.

In the following lemma, we prove that the three new bounds are better than the results in [60] in some cases.

Corollary 7.3.2 *The bounds in Theorem 7.3.1 are better than those in [60] if* $\Delta_1^2(\Delta_2^2 - \nu_2 - \Delta_2 + 1) + \Delta_1(\nu_2 - \Delta_2^2 + \Delta_2 - 1) > 0$*, or* $(\Delta_1^2 - \nu_1 - \Delta_1 + 1)\Delta_2^2 + (\nu_1 - \Delta_1^2 + \Delta_1 - 1)\Delta_2 > 0$ *or* $\Delta_1^2\Delta_2^2 - \nu_1\nu_2 + \nu_1 + \nu_2 - 1 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 + \Delta_1\Delta_2 > 0$.

Proof. Since $\Delta_1^2\Delta_2^2 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 + 3\Delta_1\Delta_2 - (\Delta_1^2\nu_2 - \Delta_1^2 - \Delta_1\nu_2 + 2\Delta_1\Delta_2 + \Delta_1) = \Delta_1^2(\Delta_2^2 - \nu_2 - \Delta_2 + 1) + \Delta_1(\nu_2 - \Delta_2^2 + \Delta_2 - 1)$, the first bound in Theorem 7.3.1 is better than that in [60] if $\Delta_1^2(\Delta_2^2 - \nu_2 - \Delta_2 + 1) + \Delta_1(\nu_2 - \Delta_2^2 + \Delta_2 - 1) > 0$.

Since $\Delta_1^2\Delta_2^2 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 + 3\Delta_1\Delta_2 - (\Delta_2^2\nu_1 - \Delta_2^2 - \Delta_2\nu_1 + 2\Delta_1\Delta_2 + \Delta_2) = (\Delta_1^2 - \nu_1 - \Delta_1 + 1)\Delta_2^2 + (\nu_1 - \Delta_1^2 + \Delta_1 - 1)\Delta_2$, our bound is better $(\Delta_1^2 - \nu_1 - \Delta_1 + 1)\Delta_2^2 + (\nu_1 - \Delta_1^2 + \Delta_1 - 1)\Delta_2 > 0$.

Since $\Delta_1^2\Delta_2^2 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 + 3\Delta_1\Delta_2 - ((\nu_1\nu_2 - \nu_1 - \nu_1 + 1) + \Delta_1\Delta_2\} + \Delta_1\Delta_2) = \Delta_1^2\Delta_2^2 - \nu_1\nu_2 + \nu_1 + \nu_2 - 1 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 + \Delta_1\Delta_2$, our bound is better $\Delta_1^2\Delta_2^2 - \nu_1\nu_2 + \nu_1 + \nu_2 - 1 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 + \Delta_1\Delta_2 > 0$. ■

7.4 The Strong Product of Graphs

In [60], upper bounds on $\lambda(G \boxtimes H)$ in terms of the maximum degree $\Delta(G \boxtimes H)$ of $G \boxtimes H$ for any two graphs G and H were obtained. In this section, we get some new results.

Theorem 7.4.1 *Let Δ_1 and Δ_2 be maximum degrees of G and H, respectively. Let ν_1 and ν_2 be the numbers of vertices of G and H, respectively. Then $\lambda(G \boxtimes H) \leq \min\{\Delta_1^2\nu_2 + \nu_2 - 1, \Delta_1^2\nu_2 + \nu_2 - 1, \nu_1\nu_2 + \Delta_1\Delta_2 - 1\}$.*

Proof. We use the Algorithm 2.1.1 to find an $L(2,1)$-labeling of $G \boxtimes H$. Let $x = (u, v) \in V(G) \times V(H)$ be a vertex with the largest label k. Note that $deg_{G\boxtimes H}(x) = deg_G(u) + deg_H(v) + deg_G(u)deg_H(v)$. Then $d = \deg_{G\boxtimes H}(x)$, $d_1 = \deg_G(u)$, $d_2 = \deg_H(v)$, $\Delta_1 = \Delta(G)$ and $\Delta_2 = \Delta(H)$. Hence, $d = d_1 + d_2 + d_1d_2$ and $\Delta = \Delta(G \boxtimes H) = \Delta_1 + \Delta_2 + \Delta_1\Delta_2$.

Let the number of vertices of G and H be ν_1 and ν_2, respectively. The adjacency matrix of $G \boxtimes H$ can be expressed as $A = A_1 \otimes A_2 + A_1 \otimes I_2 + I_1 \otimes A_2$, where A_1 and A_2 are adjacency matrices of G and H respectively, J_2 is the square matrix of order ν_2 all of whose entries are equal to 1 and I_1 is the identity matrix of order ν_1.

By [60], for fixed vertex (u_i, v_j) in $G \boxtimes H$, the number of nonzero entries in the (u_i, v_j)th row of $A^2 + A$ excluding the diagonal entries is the same as the number of nonzero entries in the (u_i, v_j)th row of $A_1^2 \otimes A_2^2 + A_1^2 \otimes I_2 + 2A_1 \otimes A_2 + I_1 \otimes A_2^2 + 2A_1^2 \otimes A_2 + 2A_1 \otimes A_2^2 + A_1 \otimes A_2 + A_1 \otimes I_2 + I_1 \otimes A_2$ excluding the diagonal entries.

Observe that the number of nonzero entries in this row excluding the diagonal entries is number of vertices at distance at most 2 from x; this number is at most the minimum of the following three:

i) $\deg_G(u_i)(\Delta_1 - 1)(\nu_2 - 1 - \deg_H(v_j)) + \deg_G(u_i)(\Delta_1 - 1) + \deg_G(u_i)\deg_H(v_j) + (\nu_2 - 1 - \deg_H(v_j)) + \deg_G(u_i)(\Delta_1 - 1)\deg_H(v_j) + \deg_G(u_i)(\nu_2 - 1 - \deg_H(v_j)) + \deg_G(u_i)\deg_H(v_j) + \deg_G(u_i) + \deg_H(v_j) \leq \Delta_1(\Delta_1 - 1)(\nu_2 - 1) + \Delta_1(\Delta_1 - 1) + \Delta_1\Delta_2 + (\nu_2 - 1) + \Delta_1(\Delta_1 - 1)\Delta_2 + \Delta_1(\nu_2 - 1) + \Delta_1\Delta_2 + \Delta_1 + \Delta_2 = \Delta_1^2\nu_2 + (\Delta_1^2 + \Delta_1 + 1)\Delta_2 + (\nu_2 - 1)$.

ii) $(\nu_1 - 1 - \deg_G(u_i))\deg_H(v_j)(\Delta_2 - 1) + (\nu_1 - 1 - \deg_G(u_i)) + \deg_G(u_i)\deg_H(v_j) + \deg_H(v_j)(\Delta_2 - 1) + (\nu_1 - 1 - \deg_G(u_i))\deg_H(v_j) + \deg_G(u_i)\deg_H(v_j)(\Delta_2 - 1) + \deg_G(u_i)\deg_H(v_j) + \deg_G(u_i) + \deg_H(v_j) \leq (\nu_1 - 1)\Delta_2(\Delta_2 - 1) + (\nu_1 - 1) + \Delta_1\Delta_2 + \Delta_2(\Delta_2 - 1) + (\nu_1 - 1)\Delta_2 + \Delta_1\Delta_2(\Delta_2 - 1) + \Delta_1\Delta_2 + \Delta_1 + \Delta_2 = \Delta_2^2\nu_1 + (\Delta_2^2 + \Delta_2 + 1)\Delta_1 + (\nu_1 - 1)$.

iii) $(\nu_1 - 1 - \deg_G(u_i))(\nu_2 - 1 - \deg_H(v_j)) + (\nu_1 - 1 - \deg_G(u_i)) + \deg_G(u_i)\deg_H(v_j) + (\nu_2 - 1 - \deg_H(v_j)) + (\nu_1 - 1 - \deg_G(u_i))\deg_H(v_j) + \deg_G(u_i)(\nu_2 - 1 - \deg_H(v_j)) + \deg_G(u_i)\deg_H(v_j) + \deg_G(u_i) + \deg_H(v_j) \leq (\nu_1 - 1)(\nu_2 - 1) + (\nu_1 - 1) + \Delta_1\Delta_2 + (\nu_2 - 1) + (\nu_1 - 1)\Delta_2 + \Delta_1(\nu_2 - 1) + \Delta_1\Delta_2 + \Delta_1 + \Delta_2 = \nu_1\nu_2 - 1 + 2\Delta_1\Delta_2 + \nu_1\Delta_2 + \Delta_1\nu_2$.

Thus,

$|I_2| + |I_1| \leq \min\{\Delta_1^2\nu_2 + (\Delta_1^2 + \Delta_1 + 1)\Delta_2 + (\nu_2 - 1), \Delta_2^2\nu_1 + (\Delta_2^2 + \Delta_2 + 1)\Delta_1 + (\nu_1 - 1), \nu_1\nu_2 - 1 + 2\Delta_1\Delta_2 + \nu_1\Delta_2 + \Delta_1\nu_2\} + \Delta_1 + \Delta_2 + \Delta_1\Delta_2$. ∎

In [60], it was proven that $\lambda(G \boxtimes H) \leq \Delta_1^2\Delta_2^2 + \Delta_1^2 + \Delta_2^2 + \Delta_1\Delta_2$. In the following lemma, we will prove that the three new bounds are better than the results in [60] in some cases.

Corollary 7.4.2 *The bounds in Theorem 7.4.1 are better than those in [60] if* $\Delta_1^2(\Delta_2^2 - \nu_2 + 2) + \Delta_2^2 - \Delta_1\Delta_2 - 2\Delta_2 - \nu_2 - \Delta_1 + 1 > 0$, *or* $\Delta_2^2(\Delta_1^2 - \nu_1 + 2) + \Delta_1^2 - \Delta_1\Delta_2 - 2\Delta_1 - \nu_1 - \Delta_2 + 1 > 0$ *or* $\Delta_1^2\Delta_2^2 + \Delta_1^2 + \Delta_2^2 - \nu_1\nu_2 + 1 - 2\Delta_1\Delta_2 - \nu_1\Delta_2 - \Delta_1\nu_2 - \Delta_1 - \Delta_2 > 0$.

Proof. Since $\Delta_1^2\Delta_2^2+\Delta_1^2+\Delta_2^2+\Delta_1\Delta_2-(\Delta_1^2\nu_2+(\Delta_1^2+\Delta_1+1)\Delta_2+(\nu_2-1)+\Delta_1+\Delta_2+\Delta_1\Delta_2) = \Delta_1^2(\Delta_2^2-\nu_2+2)+\Delta_2^2-\Delta_1\Delta_2-2\Delta_2-\nu_2-\Delta_1+1$, the first bound in Theorem 7.4.1 is better than that in [60] if $\Delta_1^2(\Delta_2^2-\nu_2+2)+\Delta_2^2-\Delta_1\Delta_2-2\Delta_2-\nu_2-\Delta_1+1 > 0$.

Since $\Delta_1^2\Delta_2^2+\Delta_1^2+\Delta_2^2+\Delta_1\Delta_2-(\Delta_2^2\nu_1+(\Delta_2^2+\Delta_2+1)\Delta_1+(\nu_1-1)+\Delta_1+\Delta_2+\Delta_1\Delta_2) = \Delta_2^2(\Delta_1^2-\nu_1+2)+\Delta_1^2-\Delta_1\Delta_2-2\Delta_1-\nu_1-\Delta_2+1$, our bound is better if $\Delta_2^2(\Delta_1^2-\nu_1+2)+\Delta_1^2-\Delta_1\Delta_2-2\Delta_1-\nu_1-\Delta_2+1 > 0$.

Since $\Delta_1^2\Delta_2^2+\Delta_1^2+\Delta_2^2+\Delta_1\Delta_2-(\nu_1\nu_2-1+2\Delta_1\Delta_2+\nu_1\Delta_2+\Delta_1\nu_2+\Delta_1+\Delta_2+\Delta_1\Delta_2) = \Delta_1^2\Delta_2^2+\Delta_1^2+\Delta_2^2-\nu_1\nu_2+1-2\Delta_1\Delta_2-\nu_1\Delta_2-\Delta_1\nu_2-\Delta_1-\Delta_2$, our bound is better if $\Delta_1^2\Delta_2^2+\Delta_1^2+\Delta_2^2-\nu_1\nu_2+1-2\Delta_1\Delta_2-\nu_1\Delta_2-\Delta_1\nu_2-\Delta_1-\Delta_2 > 0$. ■

Chapter 8

$L(2,1)$-Labelings of *Mycielski* Graphs

Mycielski graphs are an important class of graphs with interesting properties and they have been extensively studied in several coloring problems ([8], [67]). Jan Mycielski [43] created this kind of graphs to show the existence of triangle-free graphs with arbitrarily large vertex chromatic number. *Mycielski* graphs are customarily used as benchmarks for testing coloring algorithms as due to their special topology this kind of graphs contains hard to color instances [41].

In this chapter, we determine the exact value for the $L(2,1)$-labeling number of a particular class of *Mycielski* graphs, $\mu(K_n)$, where K_n is the complete graph with n vertices. We also provide, both, lower and upper bounds for the $L(2,1)$-labeling number of any *Mycielski* graph.

8.1 $L(2,1)$-Labelings of $\mu(K_n)$

In this section, we get some results on the $L(2,1)$-labeling number of *Mycielski* graphs derived from complete graphs.

Theorem 8.1.1 $\lambda(\mu(K_n)) = 2n + \lceil n/2 \rceil - 2$, *for* $n > 2$; $\lambda(\mu(K_2)) = 4$.

Proof. Let $V = V(K_n) = \{v_1, \cdots, v_n\}$ and $U = \{u_1, \cdots, u_n\}$. In $\mu(K_n)$, vertex u_i is adjacent to all vertices v_j, $j \neq i$ and vertex w is adjacent to all vertices u_i. Arrange the vertices as follows.

1) if n is even, $v_1, u_1, u_2, v_2, v_3, u_3, u_4, v_4, ..., v_{n-1}, u_{n-1}, u_n, v_n$.

2) if n is odd, $v_1, u_1, u_2, v_2, v_3, u_3, u_4, v_4, ..., v_{n-2}, u_{n-2}, u_{n-1}, v_{n-1}, v_n, u_n$.

We give vertices v_1, u_1, u_2, and v_2 labels 0, 1, 2, 3, respectively. Then we skip label 4 and give vertices v_3, u_3, u_4, and v_4 labels 5, 6, 7, 8 and so on. This is a valid labeling as v_i and u_i are at distance 2 and so are u_i and u_{i+1}. For n even, the number of labels used is $2n + n/2 - 1$ and for n odd, the number of labels used is $2n + \lceil n/2 \rceil - 1$. Vertex w gets label 4, so if $n > 2$ the largest label used is $2n + \lceil n/2 \rceil - 2$. Note that in the above labeling, the number of skipped labels (including the label for vertex w) is $\lceil n/2 \rceil - 1$.

We now prove that the above labeling is optimal. Let f be a valid $L(2,1)$-labeling for $\mu(K_n)$. For the time being we will ignore vertex w. Without loss of generality we can assume that the vertices v_i are indexed in increasing order of label, so for vertices v_i, v_j with $i < j$ it must be that $f(v_i) < f(v_j)$.

Let vertex v_i have label $f(v_i) > 0$. Since v_i is adjacent to all vertices u_j and v_j for which $j \neq i$, then only u_i is at distance 2 from v_i and hence at most one of the neighboring labels $f(v_i) - 1$ and $f(v_i) + 1$ can be used on vertex u_i. This means that at least one of the labels $f(v_i) - 1$, $f(v_i) + 1$ must be unused by labeling f.

To give a lower bound on the total number of labels that remain unused by f we associate with each pair of vertices

- $(v_2, v_3), (v_4, v_5), ..., (v_{n-2}, v_{n-1})$ if n is even, or
- $(v_2, v_3), (v_4, v_5), ..., (v_{n-1}, v_n)$ if n is odd

a unique unused label in f as follows. Note that, as shown above, for any pair of vertices (v_i, v_{i+1}) with $i < n - 1$, at least one of the labels $f(v_i) - 1$, $f(v_i) + 1$, $f(v_{i+1}) - 1$, $f(v_{i+1}) + 1$ must be unused in f, and for pair (v_{n-1}, v_n) at least one of the labels $f(v_{n-1})-1$, $f(v_{n-1})+1$, $f(v_n)-1$ must be unused in f. We associate to pair (v_i, v_{i+1}) the smallest unused label from the above sets. It is not hard to verify that each pair (v_i, v_{i+1}) gets associated a unique unused label. To see this note that the only common label that could be associated to two pairs (v_i, v_{i+1}) and (v_{i+2}, v_{i+3}) is $f(v_{i+1})+1$, if $f(v_{i+1})+1 = f(v_{i+2})-1$. To show that $f(v_{i+1})+1$ cannot be associated to pair (v_i, v_{i+1}) we consider two cases.

1) $f(v_{i+1}) = f(v_i)+2$. In this case label $f(v_i)+1$ must be unused, since it cannot be used to label any vertex u_h (because every vertex u_h is either adjacent to v_i or v_{i+1}) or any vertex v_h (because every vertex v_h is adjacent to v_i). Since $f(v_i)+1 < f(v_{i+1})+1$, then $f(v_{i+1})+1$ cannot be associated to pair (v_i, v_{i+1}).

2) $f(v_{i+1}) \geq f(v_i)+3$. In this case one of the labels $f(v_i)-1$, $f(v_i)+1$, $f(v_i)+2$ must be unused and all these labels are smaller than $f(v_{i+1})+1$.

Hence, for n even the number of unused labels in f is at least $n/2-1$ and, thus, f must use at least $2n+n/2-1$ different labels. One of the unused labels can be assigned to w.

For n odd, the number of unused labels in f is at least $(n-1)/2 = \lceil n/2 \rceil - 1$ and, thus, f must use at least $2n + \lceil n/2 \rceil - 1$ labels. One of the unused labels can be assigned to w. Thus, our labeling is optimal. ■

8.2 $L(2,1)$-Labelings of $\mu^t(K_n)$

In this section we consider the iterated Mycielski graphs $\mu^t(K_n)$, for $n \geq 2$ and $t \geq 1$.

Lemma 8.2.1 *Let z_t be the number of vertices in $\mu^t(K_n)$, for $n \geq 2$, then $z_t = 2^t(n+1)-1$.*

Proof. We prove the lemma by induction on t. If $t=1$, the conclusion holds as $z_1 = 2n+1 = 2(n+1)-1$. Assume that the conclusion holds for $t=k\geq 1$; then

$z_{k+1} = 2z_k + 1 = 2(2^k(n+1) - 1) + 1 = 2^{k+1}(n+1) - 1.$ ■

Lemma 8.2.2 *Let G be a diameter two graph. Then $\mu(G)$ is a diameter two graph.*

Proof. Let G be a graph having vertex set $V = \{v_1, \cdots, v_n\}$ and $\mu(G)$ be the Mycielski graph obtained from G by adding to it vertices $U = \{u_1, \cdots, u_n\}$ and w, and edges $\{(u_i, w), (u_i, v_j) | i, j = 1, ..., n, i \neq j\}$. By construction, any two vertices in U are at distance two from each other and w is adjacent to every vertex in U. Therefore, since G is a diameter two graph, we only need to prove that any vertex from V is at distance at most 2 from w and from any vertex in U.

We first prove that any vertex $v_k \in V$ is at distance at most 2 from any vertex $u_j \in U$. We consider two cases:

(1) Case 1. Vertices v_j and v_k are adjacent in G. Then, by construction u_j and v_k are adjacent in $\mu(G)$.

(2) Case 2. Vertices v_j and v_k are not adjacent in G. Then, there must be a vertex v_p in G such that both v_j and v_k are adjacent to v_p. By construction, u_j is adjacent to v_p and thus u_j and v_k are at distance 2 in $\mu(G)$.

We now prove that w is at distance 2 from any vertex $v_j \in V$. Let v_k be a vertex adjacent to v_j in G; then, by construction u_k is adjacent to v_j in $\mu(G)$. But u_k is also adjacent to w. Thus, w is at distance 2 from any vertex in V. ■

Corollary 8.2.3 *$\mu^t(K_n)$ is a diameter two graph, for any $t \geq 1$.*

Proof. By Lemma 8.2.2, the conclusion holds. ■

Let $V = V(K_n) = \{v_1, \cdots, v_n\}$. Let $V_k^0 = \{v_k\}$, for each $k = 1, 2, ..., n$. To construct $\mu(K_n)$ from K_n we create a copy u_i of each vertex v_i of K_n; let $V_k^1 = \{u_i\}$. To construct $\mu^2(K_n)$ we need to make copies u_i', v_i' of u_i and v_i for each $i = 1, 2, ..., n$; let $V_k^2 = \{u_i', v_i'\}$. In general, to construct $\mu^t(K_n)$ from $\mu^{t-1}(K_n)$, we need to make a set V_k^t of copies of all the vertices in $\bigcup_{i=0}^{t-1} V_k^i$.

All vertices in the sets $V_k^0, V_k^1, ..., V_k^t$ are called *copies* of v_i. The vertices in V_k^i are called the *i-th copies* of v_k. In $\mu(K_n)$ there are 2 copies of v_i, in $\mu^2(K_n)$ there are 2^2 copies of v_i, and in $\mu^t(K_n)$ there are 2^t copies of v_i. We define n disjoint groups $P_1^t, P_2^t, ..., P_n^t$ of vertices by placing in group P_k^t all copies of v_k in $\mu^t(K_n)$.

Furthermore, to construct $\mu(K_n)$ from K_n we need to create a new vertex w; let $W^1 = \{w\}$. To construct $\mu^2(K_n)$ we need to make a copy w' of w and create a new vertex w''; let $W^2 = \{w', w''\}$. To construct $\mu^t(K_n)$ from $\mu^{t-1}(K_n)$, we need to make a set W^t of copies of $\bigcup_{i=0}^{t-1} W^i$ and add to W^t another copy of w. We call the vertices in W^i the *i-th copies* of w. Let T^t be the subgraph induced in $\mu^t(G)$ by $V(\mu^t(G))/V(\mu^{t-1}(G))$; then T^t is a tree with one maximum degree vertex and in which all other vertices have degree 1. When building $\mu^t(G)$ from $\mu^{t-1}(G)$, the vertex with maximum degree in T^t is called the *last copy* of w and it is denoted as w^t. Let $W_t = \bigcup_{i=0}^{t} W^i$.

Lemma 8.2.4 *Any two vertices in a group P_k^t are at distance two in $\mu^t(K_n)$, for $n \geq 2$ and $t \geq 1$.*

Proof. We prove the lemma by induction on t. If $t = 1$, the claim holds because of the way in which $\mu(G)$ is defined. Assume that the claim holds for $t = j \geq 1$; then, in $\mu^j(K_n)$ any two vertices in a group P_k^t are at distance 2 from each other. For $\mu^{j+1}(K_n)$, we only need to consider the set V_k^{j+1} formed by the last 2^j copies of v_k. By construction, no two vertices in V_k^{j+1} are adjacent and, furthermore, since by induction hypothesis every copy of v_k in $\mu^j(K_n)$ is at distance 2 from each other, then a vertex in V_k^{j+1} cannot be adjacent to any copy of v_k in $\mu^j(K_n)$. Since by Corollary 8.2.3, $\mu^{j+1}(K_n)$ is a diameter two graph, the claim holds. ■

Lemma 8.2.5 *The number of copies of w in $\mu^t(K_n)$, for $n \geq 2$, is $2^t - 1$. Moreover, there is a feasible $L(2,1)$-labeling of $\mu^t(K_n)$ where w and its copies use consecutive labels.*

Proof. The proof is by induction on t. If $t = 1$ the conclusion trivially holds as there is only one copy of w. Assume that the lemma holds for $t = k \geq 1$, then the number of copies of w in $\mu^{k+1}(K_n)$ is $2(2^k - 1) + 1 = 2^{k+1} - 1$. Since the most recent copies $W^{k+1}/\{w^{k+1}\}$ of w are at distance two from each other, they can be labelled using consecutive labels as follows. By induction hypothesis, all copies of w in $\mu^k(K_n)$ can be labelled using consecutive labels. Let f_k be a labeling function as above for the copies of w in $\mu^k(K_n)$ and let w_a and w_b be two copies of w in $\mu^k(K_n)$ with largest and smallest labels in f_k, respectively. Then, all copies of w in $\mu^{k+1}(K_n)$ can be labelled using consecutive labels in the following way: First label, starting with 0, the $(k+1)$-th copies $W^{k+1}/\{w^{k+1}\}$ of w beginning with the $(k+1)$-th copy of w_a and ending with the $(k+1)$-th copy of w_b. Then, label all copies of w in $\mu^k(K_n)$ beginning with w_b and ending with w_a. Finally, label w^{k+1}. Note that this is a feasible labeling since w_b and the $(k+1)$-th copy of w_b are not adjacent in $\mu^{k+1}(K_n)$, also w^{k+1} is not adjacent to w_a. ■

Lemma 8.2.6 *Any vertex from $V(K_n)$ is at distance 2 from any vertex in W_t in $\mu^t(K_n)$, for $n \geq 2$ and $t \geq 1$.*

Proof. We prove the lemma by induction on t. If $t = 1$, the claim holds because of the way in which $\mu(G)$ is defined. Assume that the claim holds for $t = j \geq 1$; then, in $\mu^j(K_n)$ any vertex from $V(K_n)$ is at distance 2 from any vertex in $\bigcup_{i=0}^{j} W^i$. For $\mu^{j+1}(K_n)$, we only need to consider W^{k+1}, the $(k+1)$-th copies of w and w^{k+1}. By construction, w^{k+1} is not adjacent to any vertices in $V(K_n)$; moreover, any vertex from $W^{k+1}/\{w^t\}$ cannot be adjacent to any vertex in $V(K_n)$ as in $\mu^j(K_n)$ every vertex from $V(K_n)$ is at distance 2 from any vertex in $\bigcup_{i=0}^{j} W^i$. Since by Corollary 3.3, $\mu^{j+1}(K_n)$ is a diameter two graph, the claim holds. ■

Theorem 8.2.7 $2^t(n+1) - 2 \leq \lambda(\mu^t(K_n)) \leq 2^t(n+1) - 1 - (\lfloor n/2 \rfloor - (2^t - 1)) - 1$, *for* $2^t - 1 \leq \lfloor n/2 \rfloor$ *and* $n \geq 2$. $\lambda(\mu^t(K_n)) = 2^t(n+1) - 2$, *for* $2^t - 1 > \lfloor n/2 \rfloor$ *and* $n \geq 2$.

Proof. Let $\overrightarrow{P_k^t}$ be a list of the vertices in group P_k^t in which the vertex in V_k^0 appears at the beginning of the list and the rest of the vertices in P_k^t appear in any order; let $\overleftarrow{P_k^t}$ be the list of vertices in P_k^t in reverse order as $\overrightarrow{P_k^t}$.

By Lemma 8.2.5, the number of copies of w in $\mu^t(K_n)$ is $2^t - 1$ and we can label w and its copies using consecutive labels. Let the vertices in $W^t/\{w^t\}$ be $w_1, w_2, \ldots, w_{2^t-2}$ indexed in the same order as in the proof of Lemma 8.2.5. By Lemma 8.2.6, any vertex in $V(K_n)$ is at distance two in $\mu^t(K_n)$ from any vertex in W_t. If $2^t - 1 \leq \lfloor n/2 \rfloor$, we arrange all vertices in $\mu^t(K_n)$ as follows:

1) if n is even, $w^t\overrightarrow{P_1^t}\overleftarrow{P_2^t}w_1\overrightarrow{P_3^t}\overleftarrow{P_4^t}w_2\ldots\overrightarrow{P_{2^{t+1}-5}^t}\overleftarrow{P_{2^{t+1}-4}^t}w_{2^t-2}\ldots\overrightarrow{P_{n-1}^t}\overleftarrow{P_n^t}$

2) if n is odd, $w^t\overrightarrow{P_1^t}\overleftarrow{P_2^t}w_1\overrightarrow{P_3^t}\overleftarrow{P_4^t}w_2\ldots\overrightarrow{P_{2^{t+1}-5}^t}\overleftarrow{P_{2^{t+1}-4}^t}w_{2^t-2}\ldots\overleftarrow{P_{n-1}^t}\overrightarrow{P_n^t}$

If $2^t - 1 > \lfloor n/2 \rfloor$, we arrange the vertices in $\mu^t(K_n)$ as follows:

1) if n is even, $w^t\overrightarrow{P_1^t}\overleftarrow{P_2^t}w_1\overrightarrow{P_3^t}\overleftarrow{P_4^t}w_2\ldots\ldots\overrightarrow{P_{n-1}^t}\overleftarrow{P_n^t}w_{n/2}w_{n/2+1}w_{n/2+2}\ldots w_{2^t-2}$

2) if n is odd, $w^t\overrightarrow{P_1^t}\overleftarrow{P_2^t}w_1\overrightarrow{P_3^t}\overleftarrow{P_4^t}w_2\ldots\overrightarrow{P_{n-2}^t}\overleftarrow{P_{n-1}^t}w_{\lfloor n/2\rfloor}\overrightarrow{P_n^t}w_{\lfloor n/2\rfloor+1}w_{\lfloor n/2\rfloor+2}\ldots w_{2^t-2}$

We give vertex w^t label 0 and then use consecutive labels starting at 1 to label the vertices in $\overrightarrow{P_1^t}\overleftarrow{P_2^t}$. Notice that this is a valid labeling because of the way in which the vertices in $\overrightarrow{P_1^t}\overleftarrow{P_2^t}$ have been ordered. We assign the next available label to the copy of w placed between $\overleftarrow{P_2^t}$ and $\overrightarrow{P_3^t}$, or we skip such a label if there is no copy of w between $\overleftarrow{P_2^t}$ and $\overrightarrow{P_3^t}$. Then, we label the vertices in $\overrightarrow{P_3^t}\overleftarrow{P_4^t}$ and so on. If $2^t - 1 > \lfloor n/2 \rfloor$ the copies of w after the last vertex of $\overleftarrow{P_n^t}$ for n even or the last vertex of $\overrightarrow{P_n^t}$ for n odd, are given consecutive labels. This is a valid labeling as by Lemma 8.2.4 any two vertices in P_k^t are at distance 2 from each other and so are the last vertex of $\overrightarrow{P_k^t}$ and the first vertex of $\overleftarrow{P_{k+1}^t}$; furthermore, by Lemma 8.2.6 no copy of w is adjacent to any vertex in V_k^0 for any k. Note that by Lemma 8.2.1, the number of vertices in $\mu^t(K_n)$, for $n \geq 2$ is $2^t(n+1) - 1$. For n even, if $2^t - 1 \leq \lfloor n/2 \rfloor$, the number of labels used is $2^t(n+1) - 1 - (n/2 - (2^t - 1))$ and if $2^t - 1 > \lfloor n/2 \rfloor$, the number of labels used is $2^t(n+1) - 1$. For n odd, if $2^t - 1 \leq \lfloor n/2 \rfloor$, the number of labels used is $2^t(n+1) - 1 - (\lfloor n/2 \rfloor - (2^t - 1))$ and if $2^t - 1 > \lfloor n/2 \rfloor$, the number of labels used is $2^t(n+1) - 1$.

By Corollary 8.2.3, $\mu^t(K_n)$ is a diameter two graph, for any $t \geq 1$. Thus, any two vertices in $\mu^t(K_n)$ must have different labels. But by Lemma 8.2.1, the number of vertices in $\mu^t(K_n)$, for $n \geq 2$ is $2^t(n+1)-1$, then $\lambda(\mu^t(K_n)) \geq 2^t(n+1)-2$. And the conclusion follows. ■

8.3 $L(2,1)$-Labelings of any *Mycielski* Graphs

Let the chromatic number of G be $\chi(G) = p$ and let ν_G denote the number of vertices in G. Since $\chi(G) = p$, there is a partition $T_1, T_2, ..., T_p$ of the vertices of G such that no two vertices in T_k are adjacent and for any two different sets T_j, T_k, there are at least two adjacent vertices belonging one to T_j and the other to T_k. Let $V = V(G) = \{v_1, \cdots, v_n\}$. Similar to Section 8.2, we can define *copies* of v_i, the i-th copies of v_k, the i-th copies of w and the *last copy* of w, w^t. We define p disjoint groups $Q_1, Q_2, ..., Q_p$ of vertices by placing all vertices in T_k and all their copies into group Q_k.

Lemma 8.3.1 *In $\mu^t(G)$, for $n \geq 2$, any two vertices in the same group Q_k are not adjacent.*

Proof. We prove it by induction on t. If $t = 1$, then the conclusion holds obviously. Suppose that the conclusion holds for $t = j \geq 1$, then in $\mu^j(G)$, for $n \geq 2$, any two vertices in the same group Q_k are not adjacent. For $\mu^{j+1}(G)$, we can only consider the last $j+1$-th copies of Q_k. For all of the last $k+1$-th and previous copies of Q_k, they are not adjacent. ■

Lemma 8.3.2 *Let w_t be the number of copies of w (including w) in $\mu^t(G)$, for $n \geq 2$, then $w_t = 2^t - 1$. Moreover, we can label w and its copies using consecutive labels.*

Proof. The proof is similar to Lemma 8.2.5. ■

Lemma 8.3.3 *For any two vertices from $V(G)$ and W, respectively, they are not adjacent in $\mu^t(G)$, for $n \geq 2$ and $t \geq 1$.*

Proof. The proof is similar to Lemma 8.2.6. ■

Theorem 8.3.4 *Let the chromatic number of G, $\chi(G) = p$ and ν_G denote the number of vertices in G, then $2(p+t) - 2 \leq \lambda(\mu_t(G)) \leq \nu_G - (\lfloor p/2 \rfloor - (2^t - 1)) - 1$, for $2^t - 1 \leq \lfloor p/2 \rfloor$ and $p \geq 2$. $2(p+t) - 2 \leq \lambda(\mu_t(G)) \leq \nu_G - 1$, for $2^t - 1 > \lfloor p/2 \rfloor$ and $p \geq 2$.*

Proof. By [44], $\chi(\mu_t(G)) = p + t$. Since the chromatic number of G, $\chi(G) = p$, we can define a function f with minimal maximum label from all vertices to the positive integers such that any two adjacent vertices have different labels and thus, we can define a function $2f - 2$ with minimal maximum label from all vertices to the nonnegative integers such that any two adjacent vertices have labels at least two apart. By the definition of $\lambda(\mu_t(G))$, the $L(2,1)$-labeling of G is a function with minimal maximum label from all vertices to the nonnegative integers such that any two adjacent vertices have labels at least two apart and any two vertices at distance two have labels at least one apart. Thus, the lower bound holds.

For the upper bound, we can give similar labeling scheme as Theorem 8.2.7 and prove that it is feasible. ■

Corollary 8.3.5 *Let $\chi(G) = n$ and ν_G denote the number of vertices in G, then there is an algorithm to $L(2,1)$-label $\mu_t(G)$ with approximation ratio $(\nu_G - (\lfloor n/2 \rfloor - (2^t - 1)) - 1)/(2n + 2t - 2)$, for $2^t \leq \lfloor n/2 \rfloor$ and $n \geq 2$ and there is an algorithm to $L(2,1)$-label $\mu_t(G)$ with approximation ratio $(\nu_G - 1)/(2n + 2t - 2)$, for $2^t > \lfloor n/2 \rfloor$ and $n \geq 2$.*

Proof. By above theorem, the conclusion follows. ■

Chapter 9

Conclusions and Discussions

Due to the large number of applications of the $L(2,1)$-labeling problem and to its theoretical significance, a large number of articles on this subject have appeared in many important journals and conferences. The problem of computing the $L(2,1)$-labeling number of a graph, also called the radio coloring problem [**?**], is NP-hard even for many particular classes of graphs like diameter 2 graphs, planar graphs, and bipartite graphs. Due to the inherent hardness of the problem, only a few results are known on $L(2,1)$-labelings of general graphs. There are several interesting open problems related to particular classes of graphs, in addition to Griggs and Yeh's conjecture on general graphs.

In Chapter 2 we study $L(2,1)$-labelings on the four standard graph products and obtain significant improvements over previously best results.

In Chapter 3 we study the $L(2,1)$-labeling number of the composition of n graphs. We show that the $L(2,1)$-labelling for the composition of n graphs is much smaller than the square of the maximum degree. As a corollary, our bound is better than the bound of [60] for the composition of two graphs $G_1[G_2]$ if $\nu_2 < \Delta_2^2 + 1$, where ν_2 and Δ_2 are the number of vertices and maximum degree of G_2 respectively.

In Chapter 4 we consider the Cartesian sum of graphs and derive, both, lower and upper bounds for the $L(2,1)$-labeling number. We use two different approaches to derive the upper bounds and both approaches improve previously known bounds. We also present new approximation algorithms for the $L(2,1)$-labeling problem on Cartesian sum graphs.

In Chapter 5 we characterize d-disk graphs for $d > 1$, and give the first upper bounds on the $L(2,1)$-labeling number for this class of graphs.

In Chapter 6 we compute upper bounds for the $L(2,1)$-labeling number of total graphs of $K_{1,n}$-free graphs, where $K_{1,n}$ is the complete bipartite graph with one vertex in one side of the partition and n in the other.

In Chapter 7 we obtain more results on $L(2,1)$-labelings of the four standard graph products.

In Chapter 8 we determine the exact value for the $L(2,1)$-labeling number of a particular class of *Mycielski* graphs, $\mu(K_n)$, where K_n is the complete graph with n vertices. We also provide, both, lower and upper bounds for the $L(2,1)$-labeling number of any *Mycielski* graph.

Some of the results presented in this thesis improve on previously published results, while some others are the first known bounds for the $L(2,1)$-labeling numbers of some classes of graphs. In the future, we will, both, work on some new problems and try to improve on previous results. Specifically, we will work on the following problems:

(1) Griggs and Yeh's conjecture which states that for any graph G with maximum degree $\Delta \geq 2$, $\lambda(G) \leq \Delta^2$. The conjecture is thought to be true as it has been proved for a few classes of graphs and the upper bound is attainable by *Moore graphs* (diameter 2 graphs with $\Delta^2 + 1$ vertices), see [22]. We want to use more refined coloring techniques to try to improve Chang and Kuo's labeling scheme [10] in order to try to prove the conjecture for more classes of graphs, or even for arbitrary graphs.

(2) $L(2,1)$-labelings on planar graphs. Because the frequency assignment problem is usually defined on the plane, $L(2,1)$-labelings on planar graphs are especially important. In order to improve the previous results, on one hand, we will try to obtain more accurate results based on the existed characterizations for planar graphs; on the

other hand, we will try to find more useful characterizations for them.

(3) $L(2,1)$-labelings on Cartesian sum graphs. The previous approaches to $L(2,1)$-labelings on Cartesian sum graphs all use ad-hoc combinatorial methods. We will try to improve the previous results through a combination of combinatorial methods and our adjacency matrix approach.

(4) Variations of the $L(2,1)$-labeling problem. We will also consider studying $L(2,1)$-labeling problems on Euclidean metric and designing parameterized algorithm-s and online algorithms for $L(2,1)$-labeling problems. To the best of our knowledge these kinds of algorithms have not been studied in the context of $L(2,1)$-labeling problems.

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